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The Space of Composants of an Indecomposable Continuum

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The family of all composants of an indecomposable continuum is studied. We investigate the equivalence relation induced on an indecomposable continuum by its partition into composants. We show that up to Borel bireducibility such an equivalence relation can be of only two types: $\mathbb{E}_0$ and $\mathbb{E}_1$, the $\mathbb{E}_0$ type being “simple” and the $\mathbb{E}_1$ type being “complicated.” As a consequence of this, we show that each

composants. In particular, it follows that there is no Borel transversal for the family of all composants. This solves an old problem in the theory of continua. We prove that all hereditarily indecomposable continua are of the complicated type, that is, they fall into the $\mathbb{E}_1$ type. We analyze the properties of being of type $\mathbb{E}_1$ or of type $\mathbb{E}_0$. We show, using effective descriptive set theory, that the first of these properties is analytic and so the second one is coanalytic. We construct examples of continua of both types; in fact, we produce a family of indecomposable continua and use it to prove that these properties are complete analytic and complete coanalytic, respectively, hence non-Borel, so they do not admit simple topological characterizations. We also use continua from this family to show that an indecomposable continuum may be of type $\mathbb{E}_1$ only because of the behavior of composants on a small subset of the continuum. This, in particular, shows that certain natural approaches to solving Kuratowski's problem on generic ergodicity of the composant equivalence relation will not work. We finish with some open problems.

Key Words: indecomposable continua; composants; Borel equivalence relations.

1. INTRODUCTION[2]

1.1. *Indecomposable Continua*

A *continuum* is a metric, compact, connected space. A continuum is called *indecomposable* if it is not the union of two proper subcontinua.

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[2] For terminology unexplained in 1.1–1.3 see 1.4.

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Indecomposable continua were first constructed by Brouwer in 1910. Since then they have been thoroughly investigated. One can find information on them in [Ku2], [N] and several survey articles, for example, [Ke] and [L]. Indecomposable continua have been discovered in many contexts in dynamical systems: see [KY] for new results, a short history, and references. Recently they came up in the study of the existence of critical points on the boundaries of Siegel disks of polynomials, see [R2].

A *composant* of an indecomposable continuum C is a maximal set any two points of which lie in a proper subcontinuum of C. Each indecomposable continuum is partitioned into disjoint composants. The study of composants is crucial in understanding the structure of indecomposable continua. In particular, one set of problems is concerned with measuring the size and complexity of the space of all composants.

Kuratowski proved that each composant is meager, so there are always uncountably many of them (see [N, 11.14 and 6.19]). Mazurkiewicz [Ma] proved that there is a perfect closed set $P \subseteq C$ which has at most one point in common with each composant. An immediate consequence of it is that there are as many composants as there are reals, that is, $2^{\aleph_0}$. A natural question to ask here is whether there is a Borel function $f: C \to \mathbb{R}$ which is such that points x and y of C lie in the same composant precisely when $f(x) = f(y)$. This would establish a *definable* bijection between composants and reals. It would also show that the space of composants is simple: it admits a definable assignment of complete invariants to its members. Classically this question was asked in an equivalent form: does there exist a Borel set $T \subseteq C$ which has precisely one point in common with each composant? Such a set T is called a Borel transversal. (The equivalence of these two formulations follows from the Arsenin–Kunugui uniformization theorem for Borel sets with K_σ sections [K, Theorem 18.18] if we only take into account that each composant is a K_σ.) This question was formulated explicitly by Mauldin [M, Problem 7.1] and Rogers [R, Question 3.4] and has been considered by continuum theorists at least since the early 1960's. A partial answer was obtained by Cook [C] who proved that a Borel transversal cannot be F_σ. (More general facts about F_σ transversals were obtained in a more recent paper by Dębski and Tymchatyn [DT].) Rogers in [R] noticed the relationship between the question on the existence of a Borel transversal and the Effros theorem, see [E], [E2]. He applied the Effros theorem to prove that certain indecomposable continua (solenoids, Knaster continua [R, Theorem 4.2, Corollaries 6.2]) carry a Borel probability measure μ which is ergodic in the sense that it assigns to each composant measure 0 and for any Borel $X \subseteq C$ which is the union of a family of composants, $\mu(X) = 0$ or $\mu(X) = 1$. Continua carrying such measures do not have Borel transversals. Rogers [R, Corollary 6.10] also proved that solenoids of pseudo-arcs do not have Borel transversals for the set of all composants. By an argument of Mauldin [M, Theorem 7.2], some partial results on the nonexistence of Borel transversals can be deduced from the work of Kuratowski [Ku] (the Knaster buckethandle continuum), Krasinkiewicz [Kr] (simple continua, for definition see [Kr]), and Emeryk [Em] (Knaster continua, solenoids, the pseudo-arc).

In order to answer the question on the existence of Borel transversals and some of its natural extensions in full generality, we propose to analyze thoroughly the complexity of the space of composants of an indecomposable continuum C. We will study the equivalence relation E_C on C induced by the partition of C into composants, that is,

$$xE_C\, y \text{ iff } x \text{ and } y \text{ lie in a proper subcontinuum of } C.$$

Note that it follows from indecomposability of C that E_C is an equivalence relation. We call this equivalence relation the *composant equivalence relation.*

1.2. *Hypersmooth Equivalence Relations*

We review here basic facts about a class of equivalence relations which turns out to be particularly important in the study of composants. An equivalence relation E on a Polish space X is called *smooth* if there exists a Borel function $f: X \to Y$, for some Polish space Y, such that xEy iff $f(x) = f(y)$. This means that one can assign in a Borel fashion complete invariants, which are members of a Polish space, to the equivalence classes of E. A folklore fact says that each compact equivalence relation is smooth. (We think of an equivalence relation E on a Polish space as a subset of $X \times X$ consisting of all pairs (x, y) with xEy.) To see this, let E be a compact equivalence relation on a Polish space X. By a standard result, the space $\mathscr{K}(X)$ of all compact subsets of X is a Polish space with the Vietoris topology. Using compactness of E, we see easily that the mapping $f(x) = [x]_E \in \mathscr{K}(X)$ is Borel. Clearly xEy precisely when $f(x) = f(y)$. An equivalence relation on a Polish space is *hypersmooth* if it is an increasing union of smooth equivalence relations. In particular, all equivalence relations which are increasing unions of compact equivalence relations are hypersmooth. The structure of hypersmooth equivalence relations was investigated extensively by Kechris and Louveau in [KL]. It was proved there that they admit a classification theorem. Before we state this theorem, we need to define a couple of examples of such equivalence relations. These are $\mathbb{E}_0$ and $\mathbb{E}_1$. Let $\mathbb{E}_0$ be the equivalence relation on $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ defined by

$$x\mathbb{E}_0\, y \text{ iff } \exists N \in \mathbb{N} \forall n > N\, x(n) = y(n), \qquad \text{for} \quad x, y \in 2^{\mathbb{N}}.$$

$\mathbb{E}_1$ is an equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ defined by

$$(x_n)\,\mathbb{E}_1\,(y_n) \Leftrightarrow \exists N \in \mathbb{N}\,\forall n > N\; x_n = y_n.$$

It is not difficult to see that both $\mathbb{E}_0$ and $\mathbb{E}_1$ are increasing unions of compact equivalence relations and so they are hypersmooth. Another hypersmooth equivalence relation that will be relevant to our study is $\mathbb{E}_0 \times 2^{\mathbb{N}}$ defined on $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ and given by the formula $(x, y)(\mathbb{E}_0 \times 2^{\mathbb{N}})(x', y')$ precisely when $x\mathbb{E}_0 x'$. We will also need a way of comparing equivalence relations. Let E and F be equivalence relations on Polish spaces X and Y, respectively. We say that E is *Borel reducible* to F, in symbols $E \leqslant_B F$, precisely when there exists a Borel function $f\colon X \to Y$ such that xEy iff $f(x)\,F f(y)$. Note that being smooth means, in this terminology, being Borel reducible to the equality equivalence relation on a Polish space. We will also write $E \approx_B F$ if $E \leqslant_B F$ and $F \leqslant_B E$. For example, one easily checks that $\mathbb{E}_0 \approx_B \mathbb{E}_0 \times 2^{\mathbb{N}}$. We write $E <_B F$ if $E \leqslant_B F$ and $F \nleqslant_B E$.

The conjunction of results of [HKL] and [KL] (see [KL] remarks following Theorem 2.1) gives that if E is a hypersmooth equivalence relation, then

$$E \text{ is smooth, or } E \approx_B \mathbb{E}_0, \text{ or } E \approx_B \mathbb{E}_1.$$

Moreover, if E is smooth, then $E <_B \mathbb{E}_0$ and $\mathbb{E}_0 <_B \mathbb{E}_1$. The inequalities $E \leqslant_B \mathbb{E}_0$, for smooth E, and $\mathbb{E}_0 \leqslant \mathbb{E}_1$ are standard and easy. The fact that $\mathbb{E}_0 \nleqslant_B E$ for any smooth equivalence relation E is standard and follows from ergodicity of the product (Lebesgue) measure on $2^{\mathbb{N}}$; $\mathbb{E}_1 \nleqslant_B \mathbb{E}_0$ has also been known for some time, see [FR] and [K2, Section 5]. A sharper result, from which $\mathbb{E}_1 \nleqslant_B \mathbb{E}_0$ follows immediately, was established in [KL, Theorem 1.5].

1.3. *Composant Equivalence Relation*

As in 1.1, we denote here by E_C the composant equivalence relation on an indecomposable continuum C. Strengthening a result of Rogers [R], we show (Corollary 2.4) that E_C is hypersmooth. We prove that $\mathbb{E}_0 \leqslant_B E_C$ so E_C is never smooth (Corollaries 3.3 and 3.4). As a consequence of it, we get that there is no Borel transversal for the set of all composants of an indecomposable continuum. Another consequence of it and of the Kechris–Louveau classification theorem is that there are only two possibilities for E_C:

$$E_C \approx_B \mathbb{E}_0 \qquad \text{or} \qquad E_C \approx_B \mathbb{E}_1.$$

Further, we show (Theorem 4.3) that

$$E_C \approx_B \mathbb{E}_1 \text{ if } C \text{ is hereditarily indecomposable.}$$

In fact, for $E_C \approx_B \mathbb{E}_1$ to hold it suffices that for some point x in C all *proper* subcontinua containing x are hereditarily indecomposable. One intuitively feels that there is a "qualitative" difference in complexity between, say, hereditarily indecomposable continua and, say, Knaster continua. The latter ones can be easily defined and even drawn, while any construction of the former is rather complex. The above result gives a firm basis to this intuition; there indeed exists a split: $E_C \approx_B \mathbb{E}_0 \,/\, E_C \approx_B \mathbb{E}_1$.

We take a closer look at this split. On the one hand, we show that the class of indecomposable subcontinua C of $[0, 1]^{\mathbb{N}}$ for which $E_C \approx_B \mathbb{E}_1$ is analytic, that is, $\mathbf{\Sigma}^1_1$ and, therefore, that the class of indecomposable continua for which $E_C \approx_B \mathbb{E}_0$ is coanalytic, $\mathbf{\Pi}^1_1$ (Theorem 5.1). The proof of this essentially topological result uses effective descriptive set theory which is rare for natural classes from topology or analysis although not unprecedented: see Becker's Example 17 from [B]. On the other hand, we exhibit a family of indecomposable continua, $\{C^z \colon z \in X\}$, parametrized continuously by a Polish space X. (That is, the mapping $X \ni z \to C^z \in \mathscr{K}([0, 1]^{\mathbb{N}})$ is continuous and each C^z is an indecomposable continuum.) The family is such that for a set of parameters $A \subseteq X$ which is $\mathbf{\Sigma}^1_1$-complete, $E_{C^z} \approx_B \mathbb{E}_1$ for $z \in A$ and $E_{C^z} \approx_B \mathbb{E}_0$ for $z \notin A$ (Theorem 5.2). In particular, since the family of hereditarily indecomposable continua is G_δ (that is, $\mathbf{\Pi}^0_2$), this indicates that the condition $E_C \approx_B \mathbb{E}_1$ is substantially different, and more complicated than, simply C being hereditarily indecomposable. In fact, it shows that the class of all indecomposable continua C for which $E_C \approx_B \mathbb{E}_1$ does not have a simple topological characterization. Also, taken together with what was said above, it shows that this class is $\mathbf{\Sigma}^1_1$-complete (Corollary 5.14).

The continua C^z will also serve a different purpose. As mentioned above, Rogers in [R] observed that the existence of an E_C-ergodic probability measure on C implies the nonexistence of a Borel transversal. Mauldin in [M, Theorem 7.2] made a similar observation except that he considered the notion of category rather than measure. He pointed out that if E_C is generically ergodic, that is, each Borel E_C-invariant subset of C is meager or comeager, then there is no Borel transversal for E_C. The question whether E_C is generically ergodic is an old problem of Kuratowski [Ku, p. 255]. (Generic ergodicity is called strict transitivity in [Ku].) In effect, by [M], the question on the existence of Borel transversals turns out to be a weaker form of Kuratowski's problem. Now, $\mathbb{E}_1$, $\mathbb{E}_0$ and $\mathbb{E}_0 \times 2^{\mathbb{N}}$ are generically ergodic. (This follows from [K, 8.47] since sets invariant with respect to these equivalence relations are tail sets in appropriate product spaces.) Thus, it is natural to attempt to solve Kuratowski's problem by proving that if $E_C \approx_B \mathbb{E}_1$, then there is a Borel isomorphism between E_C and $\mathbb{E}_1$ which preserves meagerness and, similarly, if $E_C \approx_B \mathbb{E}_0$, then such an isomorphism exists between E_C and $\mathbb{E}_0 \times 2^{\mathbb{N}}$. (We cannot require that E_C be isomorphic to $\mathbb{E}_0$ since the equivalence classes of $\mathbb{E}_0$ are countable but, on the other hand, as we saw in 1.2, $\mathbb{E}_0 \approx_B \mathbb{E}_0 \times 2^{\mathbb{N}}$.) It turns out that some of our continua C^z give counterexamples to this approach (Corollary 5.17). However, a milder version of it may still be possible, see Section 6.

1.4. *Notation*

Our notation is standard. We explain here some pieces of it in order to avoid ambiguity.

By a Polish space we mean a metric, separable, complete space. In certain situations, it is more natural to consider an "inequality" between equivalence relations which is sharper than $\leqslant_B$. If E and F are Borel equivalence relations on Polish spaces X and Y, we say that E *continuously embeds* in F, $E \sqsubseteq_c F$, if there is a continuous injection $f\colon X \to Y$ such that xEy iff $f(x)\, F f(y)$ for all $x, y \in X$. If E and F are essentially identical, we will write $E \cong F$, that is, $E \cong F$ if there exists a homeomorphism $f\colon X \to Y$ such that xEy iff $f(x)\, F f(y)$. If E is an equivalence relation on a set X, let $[x]_E$ stand for the equivalence class of $x \in X$ and $[A]_E$ for the saturation of $A \subseteq X : [A]_E = \{y \in X : \exists x \in A\, xEy\}$. We will write $\neg(xEy)$ to indicate that x and y are not in the relation E. By $2^{\mathbb{N}}$ we will denote the equivalence relation on the space $2^{\mathbb{N}}$ whose only equivalence class is the whole space.

Let X be a Polish space. Recall that a subset of X is *analytic*, that is, $\mathbf{\Sigma}_1^1$, if it is the projection of a Borel subset of $X \times \mathbb{N}^{\mathbb{N}}$ or, equivalently, if it is the continuous image of a Borel subset of $\mathbb{N}^{\mathbb{N}}$. If X is uncountable, the class of $\mathbf{\Sigma}_1^1$ sets properly contains the class of all Borel sets. A set $A \subseteq X$ is called $\mathbf{\Sigma}_1^1$-*hard* if for any $\mathbf{\Sigma}_1^1$ subset B of $\mathbb{N}^{\mathbb{N}}$ there exists a continuous function $f\colon \mathbb{N}^{\mathbb{N}} \to X$ such that $B = f^{-1}(A)$. Note that sets which are $\mathbf{\Sigma}_1^1$-hard are not Borel. A set $A \subseteq X$ is $\mathbf{\Sigma}_1^1$-*complete* if it is both $\mathbf{\Sigma}_1^1$ and $\mathbf{\Sigma}_1^1$-hard. Complements of $\mathbf{\Sigma}_1^1$ sets are called $\mathbf{\Pi}_1^1$. Borel sets turn out to be precisely those sets which are both $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$. The class of Borel sets is denoted by $\mathbf{\Delta}_1^1$. Projections of $\mathbf{\Pi}_1^1$ subsets of $X \times \mathbb{N}^{\mathbb{N}}$ are called $\mathbf{\Sigma}_2^1$ and complements of $\mathbf{\Sigma}_2^1$ are called $\mathbf{\Pi}_2^1$. These classes are much bigger than $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$; for example, their intersection properly contains the σ-algebra generated by the union of $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$. Recall that closed, open, and G_δ sets are denoted by $\mathbf{\Pi}_1^0$, $\mathbf{\Sigma}_1^0$, $\mathbf{\Pi}_2^0$, respectively. Much more on this can be found in [K].

$\mathbb{N}$ stands for the set of all natural numbers including 0. By $\mathbb{N}^{<\mathbb{N}}$, we denote the set of all finite sequences of natural numbers. For $s \in \mathbb{N}^{<\mathbb{N}}$, let $|s|$ be the length of s, that is, the unique $n \in \mathbb{N}$ with $s \in \mathbb{N}^n$. If $s \in \mathbb{N}^{<\mathbb{N}}$ and $n \in \mathbb{N}$, let sn be the sequence of length $|s|+1$ extending s and such that $sn(|s|) = n$. For $l \leqslant |s|$, let $s \,|\, l$ be the unique element of $\mathbb{N}^l$ which is extended by s. Similarly, if $x \in \mathbb{N}^{\mathbb{N}}$, let $x \,|\, l$ be the finite sequence of length l extended by x. Quantifiers $\forall^\infty n$ and $\exists^\infty n$ mean "for all but finitely many n" and "for infinitely many n." We use them only to make certain definitions look shorter.

For a Polish space X, let $\mathcal{K}(X)$ stand for the space of all compact subsets of X with the Vietoris topology. This space is itself Polish, and if X is additionally compact, so is $\mathcal{K}(X)$ [N, 4.13]. In that case, the subset of $\mathcal{K}(X)$ consisting of all continua is compact as well [N, 4.17].

2. THE COMPOSANT EQUIVALENCE RELATION IS HYPERSMOOTH

The following theorem, improving on a result of Rogers [R, Theorem 3.3], gives an important structural property of the composant equivalence relation.

THEOREM 2.1. *The composant equivalence relation on an indecomposable continuum is the increasing union of a sequence of compact equivalence relations. Additionally, the compact equivalence relations in this sequence can be chosen so that their equivalence classes are continua.*

Proof. Let C be an indecomposable continuum with the composant equivalence relation E_C, and let d be a metric on C. Pick points z_0 and z_1 which lie in different composants of C. Let V_n, $n \in \mathbb{N}$, be the open ball centered at z_0 with radius $1/(n+1)$. Define C_n to be the component of z_0 in $C \setminus \{x: d(x, z_1) < 1/(n+1)\}$. Then C_n are proper subcontinua of C such that $z_0 \in C_n$, $C_n \subseteq C_{n+1}$, and $\bigcup_n C_n = [z_0]_{E_C} =$ the composant of z_0. Define for $x, y \in C$ and $n \in \mathbb{N}$

(1) $\quad xF_n y$ iff $x = y$ or $x, y \in K$ for some subcontinuum $K \subseteq (C \setminus V_n) \cup C_n$.

One checks easily that each F_n is an equivalence relation, that $F_n \subseteq F_{n+1}$, and that each F_n equivalence class is connected. Once we prove that each F_n is closed, this last condition will imply that each F_n equivalence class is a continuum. To see that F_n is closed, let $x_k F_n y_k$, $k \in \mathbb{N}$, and $x_k \to x$, $y_k \to y$. We can assume that $x_k \neq y_k$ for all k. Let K_k be a continuum witnessing $x_k F_n y_k$. Then $K = \lim_k K_k$ is a continuum, $x, y \in K$, and $K \subseteq (C \setminus V_n) \cup C_n$ since $(C \setminus V_n) \cup C_n$ is closed. Thus, $xF_n y$.

Since $V_n \setminus C_n \neq \varnothing$ for all n (as C_n is nowhere dense, see [Ku2]), each subcontinuum $K \subseteq (C \setminus V_n) \cup C_n$ is proper, whence $F_n \subseteq E_C$ for all $n \in \mathbb{N}$. To see that $\bigcup_n F_n = E_C$, let $xE_C y$. If $xE_C z_0$, we can find an $n \in \mathbb{N}$ such that $x, y \in C_n$. But then $xF_n y$. If $\neg(xE_C x_0)$, let K be a proper subcontinuum of C with $x, y \in K$. There is n such that $K \cap V_n = \varnothing$, as $z_0 \notin K$. Then $xF_n y$. This finishes the proof.

COROLLARY 2.2. *The composant equivalence relation on an indecomposable continuum is hypersmooth.*

Proof. By Theorem 2.1, the composant equivalence relation is an increasing union of countably many compact equivalence relations. As is well known, and was explained in the introduction (1.1), such equivalence relations are hypersmooth.

3. EMBEDDING $\mathbb{E}_0$ IN K_σ EQUIVALENCE RELATIONS AND APPLICATIONS TO E_C

We prove below a theorem which gives a sufficient condition for a K_σ equivalence relation to continuously embed $\mathbb{E}_0$. This theorem is related to and was inspired by the Glimm–Effros theorem on continuous actions of Polish groups discovered in the study of C^*-algebras [G], [E] and its generalization to actions of arbitrary groups of homeomorphisms due to Becker and Kechris [BK, Theorem 3.4.5].

THEOREM 3.1. *Let X be Polish, and let F be a K_σ equivalence relation on X. Assume that $\{x \in X : [x]_F$ is not locally closed at $x\}$ is not meager. Then $\mathbb{E}_0 \sqsubseteq_c F$.*

Proof. Since F is K_σ and contains the diagonal of $X \times X$, X is K_σ. Hence, there exist open U_n, $n \in \mathbb{N}$, such that $\bigcup_n U_n$ is dense and for each n, $\bar{U}_n$ is compact. Thus $\{x \in U_{n_0} : [x]_F$ is not locally closed at $x\}$ is not meager for some n_0. So, restricting F to $\bar{U}_{n_0}$, we can assume that X is compact.

Now, we can find $F_k \subseteq F$, $k \in \mathbb{N}$, such that F_k is compact, symmetric (i.e., $(x, y) \in F_k$ implies $(y, x) \in F_k$), $\{(x, x): x \in X\} \subseteq F_k$, and $F_k^{2k+3} \subseteq F_{k+1}$. ($F_k^n$ is defined recursively: $F_k^1 = F_k$, and $F_k^{n+1} = \{(x, y): \exists z\,(x, z) \in F_k^n$ and $(z, y) \in F_k\}$.) We write $A \perp_k B$ for $A, B \subseteq X$ if $(A \times B) \cap F_k = \varnothing$.

Claim 1. There exists an open nonempty set $U \subseteq X$ such that given $k \in \mathbb{N}$ and $\varnothing \neq W \subseteq U$ open there are nonempty compact $C_0, C_1 \subseteq W$ and $n \in \mathbb{N}$ such that

(i) $C_0 \perp_k C_1$;

(ii) $C_0 \subseteq [C_1]_{F_n}$ and $C_1 \subseteq [C_0]_{F_n}$;

(iii) $C_0 = \bar{V}$ for some open V.

Proof of Claim 1. Let $\{V_m: m \in \mathbb{N}\}$ be an open basis for X with each V_m nonempty. Put

$$A_{m,p} = ([V_m]_F \cap V_p) \cup (V_m \cap [V_p]_F).$$

Note that $A_{m,p}$ are F_σ. Put

$$B_k^r = \bigcup \{A_{m,p}\colon V_m \perp_k V_p \text{ and } V_m, V_p \subseteq V_r\}, \qquad k, r \in \mathbb{N}.$$

First, we show that if $x \notin \bigcap_k \bigcap_r (B_k^r \cup (X \setminus V_r))$, then $[x]_F$ is locally closed at x. If x is as above, then $x \in V_r$ and $x \notin B_k^r$ for some $k, r \in \mathbb{N}$. Let $y \in V_r$. Then $(x, y) \in F$ iff $(x, y) \in F_k$. Since $F_k \subseteq F$, it is enough to show that $(x, y) \notin F_k$ implies $(x, y) \notin F$. But if $(x, y) \notin F_k$, then there are $V_m, V_p \subseteq V_r$ such that $x \in V_m$, $y \in V_p$, and $V_m \perp_k V_p$. Since $x \notin B_k^r$ and $x \in V_m$, $x \notin [V_p]_F$ whence $(x, y) \notin F$. It follows that $[x]_F \cap V_r = [x]_{F_k} \cap V_r$ whence $[x]_F$ is locally closed at x.

By assumption, $\bigcap_k \bigcap_r (B_k^r \cup (X \setminus V_r))$ is not meager. Since $B_k^r \cup (X \setminus V_r)$ is F_σ, there exists a nonempty open set U such that for all r, k, $\mathrm{int}(B_k^r \cup (X \setminus V_r))$ is dense in U. Let $\emptyset \neq W \subseteq U$ be open. If $V_r \subseteq W$, then for all k, $\mathrm{int}(B_k^r)$ is dense in V_r, whence for any k there are $V_p, V_m \subseteq V_r$ such that $V_p \perp_k V_m$ and $\mathrm{int}([V_p]_F \cap V_m) \neq \emptyset$. Now, we can find $l, n \in \mathbb{N}$ such that $\bar{V}_l \subseteq V_m$ and $\bar{V}_l \subseteq [V_p]_{F_n}$. Put $C_0 = \bar{V}_l$ and $C_1 = \bar{V}_p \cap [\bar{V}_l]_{F_n}$. Then C_0 and C_1 are as required, which finishes the proof of Claim 1.

We construct recursively nonempty compact sets C_s, $s \in 2^{<\mathbb{N}}$ (as usual $C_s \subseteq C_t$ if $s \supseteq t$, and $\mathrm{diam}(C_s) \leqslant 1/(|s|+1)$) along with a sequence of natural numbers $n_0 < n_1 < n_2 < \cdots$ so that to some pairs (C_s, C_t), $s, t \in 2^k$, an n_i with $i < k$ will be assigned in which case, we write $C_s \overset{n_i}{\longleftrightarrow} C_t$. The following additional conditions will be fulfilled. (By 0^k we denote the sequence consisting of k 0's.)

(1) $C_{0^k} = \bar{U}_k$ where U_k is open;
(2) if $C_s \overset{n_k}{\longleftrightarrow} C_t$, then $C_t \subseteq [C_s]_{F_{n_k}}$ and $C_s \subseteq [C_t]_{F_{n_k}}$;
(3) $C_{0^{k+1}} \overset{n_k}{\longleftrightarrow} C_{0^k i}$ for $i = 0, 1$;
(4) $C_{0^{k+1}} \perp_{n_{k-1}+2} C_{0^k 1}$;
(5) if $C_s \overset{n_k}{\longleftrightarrow} C_t$, then $C_{si} \overset{n_k}{\longleftrightarrow} C_{ti}$ for $i = 0, 1$.

Assume the construction has been carried out.

Claim 2. $\mathbb{E}_0 \sqsubseteq_c F$.

Proof of Claim 2. Call $s, t \in 2^{<\mathbb{N}}$ k-close if $|s| = |t|$, there is $p \leqslant k$ such that $s \mid (p+1) = 0^{p+1}$, $t \mid (p+1) = 0^p 1$ or vice versa, and for any m with $p+1 \leqslant m < |s|$, $s(m) = t(m)$. Immediately from (2), (3) and (5), we get that if $s, t \in 2^{<\mathbb{N}}$ are k-close, then $[C_s]_{F_{n_k}} \supseteq C_t$ and $[C_t]_{F_{n_k}} \supseteq C_s$. Also, it is clear that if $s, t \in 2^{<\mathbb{N}}$, $|s| = |t|$, and $s(i) = t(i)$ for all $i \geqslant k+1$, then there is a sequence $s_0, s_1, \ldots, s_m$ such that $m \leqslant 2k$, $s_0 = s$, $s_m = t$, and s_i, s_{i+1} are k-close for $i < m$. Thus, if $s, t \in 2^{<\mathbb{N}}$ are as above, then $[C_s]_{F_{n_k}^{2k}} \supseteq C_t$ and $[C_t]_{F_{n_k}^{2k}} \supseteq C_s$. Since $F_{n_k}^{2k} \subseteq F_{n_k+1}$, we obtain the following conclusion.

(i) Let $s, t \in 2^{<\mathbb{N}}$, $|s| = |t|$, and $s(i) = t(i)$ for $i \geqslant k+1$. Then $[C_s]_{F_{n_k+1}} \supseteq C_t$ and $[C_t]_{F_{n_k+1}} \supseteq C_s$.

Also we have the following fact.

(ii) Let $s, t \in 2^{<\mathbb{N}}$, $|s| = |t|$. Assume $s(k) \neq t(k)$, $k \geqslant 1$. Then $C_s \perp_{n_{k-1}} C_t$. If $s(0) \neq t(0)$, then clearly $C_s \perp_0 C_t$.

To see this, assume $s(k) = 0$, $t(k) = 1$, and put $s' = s \mid (k+1)$, $t' = t \mid (k+1)$. By (i), $C_{s'} \subseteq [C_{0^{k+1}}]_{F_{n_{k-1}+1}}$ and $C_{t'} \subseteq [C_{0^k 1}]_{F_{n_{k-1}+1}}$. Now if $C_s \not\perp_{n_{k-1}} C_t$, then there are $x \in C_s$, $y \in C_t$ with $x F_{n_{k-1}} y$. Since $C_s \subseteq C_{s'}$ and $C_t \subseteq C_{t'}$, we get $z_0 \in C_{0^{k+1}}$ and $z_1 \in C_{0^k 1}$ with $(x, z_0) \in F_{n_{k-1}+1}$ and $(y, z_1) \in F_{n_{k-1}+1}$. Thus, $(z_0, z_1) \in F^3_{n_{k-1}+1} \subseteq F_{n_{k-1}+2}$ which contradicts (4).

Define $\phi\colon 2^{\mathbb{N}} \to X$ by letting $\phi(\alpha)$ be the unique element in $\bigcap_n C_{\alpha|n}$ for $\alpha \in 2^{\mathbb{N}}$. Since $\{(x, x)\colon x \in X\} \subseteq F_k$ for all k, from (ii) we get that if $|s| = |t|$ and $s \neq t$, then $C_s \cap C_t = \varnothing$. Thus, ϕ is 1-to-1 and continuous. If $\alpha, \beta \in 2^{\mathbb{N}}$ and $(\alpha, \beta) \notin \mathbb{E}_0$, that is, $\alpha(k) \neq \beta(k)$ for infinitely many $k \in \mathbb{N}$, then by (ii) and the fact that $F_k \subseteq F_{k+1}$ for all k, we have $(\phi(\alpha), \phi(\beta)) \notin F_k$ for all k whence $(\phi(\alpha), \phi(\beta)) \notin F$. If $\alpha, \beta \in 2^{\mathbb{N}}$ and $(\alpha, \beta) \in \mathbb{E}_0$, then $\alpha(k) = \beta(k)$ for $k \geqslant N$ and some $N \in \mathbb{N}$. By (ii), $[C_{\alpha|m}]_{F_{n_N+1}} \supseteq C_{\beta|m}$ and $[C_{\beta|m}]_{F_{n_N+1}} \supseteq C_{\alpha|m}$ for all k. Hence $[C_{\alpha|m}]_{F_{n_N+1}} \notin \phi(\beta)$ and $[C_{\beta|m}]_{F_{n_N+1}} \notin \phi(\alpha)$. This allows us to pick sequences $y_m \to \phi(\alpha)$ and $z_m \to \phi(\beta)$ with $y_m \in [\phi(\beta)]_{F_{n_N}+1}$ and $z_m \in [\phi(\alpha)]_{F_{n_N}+1}$. Since $[\phi(\alpha)]_{F_{n_N}+1}$ and $[\phi(\beta)]_{F_{n_N}+1}$ are closed, $\phi(\alpha) \in [\phi(\beta)]_{F_{n_N}+1}$ and $\phi(\beta) \in [\phi(\alpha)]_{F_{n_N}+1}$, whence $\phi(\alpha)\, F \phi(\beta)$, and Claim 2 is proved.

Thus, to finish the proof of the theorem, it is enough to construct $\{C_s\colon s \in 2^{<\mathbb{N}}\}$. The construction is recursive on the length of $s \in 2^{<\mathbb{N}}$. To avoid cluttering pages with notation, we will describe only the first three steps of the construction. Let U be as in Claim 1. Put $C_\varnothing = \bar{U}_0$ where U_0 is a nonempty, open set with $\operatorname{diam}(U_0) < 1$ and $\bar{U}_0 \subseteq U$. Find n_0 and D_0, D_1 as in Claim 1 for $W = U_0$ and $k = 0$. Let V be an open set with $D_0 = \bar{V}$. Let $D^1, \ldots, D^m$ be compact sets with diameter $< 1/2$ and whose union is D_1. Then for some i_0, $[D^{i_0}]_{F_{n_0}} \cap D_0$ has a nonempty interior. Let U_1 be open with diameter $< 1/2$ and such that $\bar{U}_1 \subseteq \operatorname{int}([D^{i_0}]_{F_{n_0}} \cap D_0)$. Finally, put $C_{\langle 0\rangle} = \bar{U}_1$ and $C_{\langle 1\rangle} = D^{i_0} \cap [C_0]_{F_{n_0}}$. Now, we define C_s for s with $|s| = 2$. Let n_1, D_{00}, D_{01} be as in Claim 1 for $W = U_1$ and $k = n_0 + 2$. Let V be open with $D_{00} = \bar{V}$. Put $D_{10} = C_1 \cap [D_{00}]_{F_{n_0}}$ and $D_{11} = C_1 \cap [D_{01}]_{F_{n_0}}$. We could define $C_{\langle i, j\rangle}$ to be D_{ij} except that their diameters may be too big, so in the remainder of the proof, we modify them appropriately. First, find $D^1_{11} \subseteq D_{11}$ compact with diameter $< 1/3$ and such that the interior of $[[D^1_{11}]_{F_{n_0}} \cap D_{01}]_{F_{n_1}} \cap U_1$ is nonempty. Next, find $D^1_{01} \subseteq [D^1_{11}]_{F_{n_0}} \cap D_{01}$ compact with diameter $< 1/3$ and such that the interior of $[D^1_{01}]_{F_{n_1}} \cap U_1$ is nonempty. Find D^1_{10} compact with diameter $< 1/3$ and such that the interior of $[D_{10}^1]_{F_{n_0}} \cap [D_{01}^1]_{F_{n_1}} \cap U_1$ is nonempty. Let U_2 be an open set such that $\operatorname{diam}(U_2) < 1/3$ and $\bar{U}_2 \subseteq [D_{10}^1]_{F_{n_0}} \cap [D_{01}^1]_{F_{n_1}} \cap U_1$. Put finally $C_{\langle 0,0\rangle} = \bar{U}_2, C_{\langle 1,0\rangle} = [C_{\langle 0,0\rangle}]_{F_{n_0}} \cap D_{10}^1, C_{\langle 0,1\rangle} = [C_{\langle 0,0\rangle}]_{F_{n_1}} \cap D_{01}^1$, and $C_{\langle 1,1\rangle} = [C_{\langle 0,1\rangle}]_{F_{n_0}} \cap D_{11}^1$. This finishes the proof of the theorem.

COROLLARY 3.2. *Let X be a Polish space. Let F be a K_σ equivalence relation on X each equivalence class of which is dense. If F has at least two equivalence classes, then $\mathbb{E}_0 \sqsubseteq_c F$.*

Proof. By Theorem 3.1 it is enough to show that for any $x \in X$, $[x]_F$ is not locally closed at x. But if it were, then, since $[x]_F$ is dense, there would exist an open set U with $x \in U \subseteq [x]_F$. But then no equivalence class different from $[x]_F$ could be dense.

Corollary 3.2 points to an amusing difference between K_σ equivalence relations and equivalence relations induced by continuous actions of Polish groups. (An equivalence relation is of the latter type if it is induced by the partition of a Polish space into orbits of a continuous action of a Polish group.) It was shown in [HS, Examples 4.1 or 4.4] that it is possible to have a continuous action of a Polish group on a Polish space with precisely two orbits both of which are dense. Corollary 3.2 above very strongly rules out such a possibility for K_σ equivalence relations.

Below, we answer the question on the existence of Borel transversals in the negative for all indecomposable continua. The following corollary will imply that each indecomposable continuum carries an ergodic probability measure (see Corollary 3.4).

COROLLARY 3.3. *Let C be an indecomposable continuum.*

(i) $\mathbb{E}_0 \sqsubseteq_c E_C$ *where E_C is the composant equivalence relation.*

(ii) $E_C \approx_B \mathbb{E}_0$ *or* $E_C \approx_B \mathbb{E}_1$.

Proof. (i) By [R, Theorem 3.3] (or see Theorem 2.1 in this paper), E_C is K_σ. It is well known, see for example [Ku2, Ch. 5, § 48, VI, Theorems 2 and 7], that each composant is dense and that there are at least two composants, that is, $[x]_{E_C}$ is dense for each $x \in C$ and E_C has at least two equivalence classes. Thus, Corollary 3.3 follows from Corollary 3.2.

(ii) is immediate from (i), Corollary 2.2, and the classification of hypersmooth equivalence relations [KL] (see the introduction 1.2).

To state the next corollary, we need the following definition. Let E be a Borel equivalence relation on a Polish space Y. A Borel probability measure μ on Y is called *E-ergodic* if $\mu([x]_E) = 0$ for any $x \in Y$ and $\mu(X) = 0$ or 1 if $X \subseteq Y$ is Borel and is the union of a family of E-equivalence classes. The next corollary follows from Corollary 3.3 by, bynow, standard arguments. It answers [M, Problem 7.1] and [R, Question 3.4].

COROLLARY 3.4. *Let C be an indecomposable continuum with the composant equivalence relation E_C.*

(i) *There exists an E_C-ergodic Borel probability measure on C.*

(ii) *There does not exist a Borel set which has precisely one point in common with each composant.*

(iii) *E_C is not smooth.*

Proof. (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii) are well-known and come from [E]. To see the first of these implications assume towards contradiction that E_C is smooth and let X be a Polish space and $f: C \to X$ a Borel function witnessing smoothness of E_C. Let μ be a measure as in (i). It is now easy to see that f must be constant on a set of μ-measure 1. This would mean that some E_C-equivalence class has measure 1 which contradicts E_C-ergodicity of μ. To see the second implication note that if we had a Borel subset T of C which intersects each composant in precisely one point, then the function $f: C \to C$ defined by letting $f(x)$ be the unique point in the intersection of T with the composant containing x would be Borel and would witness smoothness of E_C. (Borelness of f follows from E_C being K_σ.) Point (i) is a consequence of Corollary 3.3(i). Simply pick a continuous function $f: 2^{\mathbb{N}} \to C$ witnessing $\mathbb{E}_0 \sqsubseteq_c E_C$ and push Lebesgue measure on $2^{\mathbb{N}}$ to C using this f.

4. $\mathbb{E}_1$ AND HEREDITARILY INDECOMPOSABLE CONTINUA

In this section, we prove that if C is hereditarily indecomposable, then only the second possibility from Corollary 3.3(ii) can occur: $E_C \approx_B \mathbb{E}_1$.

We will need a certain combinatorial representation of $\mathbb{E}_1$. Let (k_n) be a sequence of natural numbers. We recursively define equivalence relations E_k, $k \in \mathbb{N}$, on $2^{<\mathbb{N}} = \bigcup_n 2^n$. (Recursion is on n.) First declare that if $s, t \in 2^{<\mathbb{N}}$ and $|s| \neq |t|$, then $\neg(sE_k t)$ for all k.

($n = 0$) Each E_k is the only possible equivalence relation on the one element set $2^0 = \{\varnothing\}$.

(n) Assume all the E_k's have been defined on $2^{<n}$. Define E_k on 2^n as follows:

1. if $k < k_n$, then

$$\forall s, t \in 2^{n-1} (\neg(s0E_k t1) \text{ and } (sE_k t \Leftrightarrow s0E_k t0 \text{ and } s1E_k t1));$$

2. if $k \geqslant k_n$, then

$$\forall s, t \in 2^{n-1} (sE_k t \Leftrightarrow s0E_k s1E_k t0E_k t1).$$

Define an equivalence relation on $2^{\mathbb{N}}$ by

$$xE_{(k_n)}\, y \Leftrightarrow \exists k\, \forall n\, x \,|\, nE_k\, y \,|\, n.$$

The following lemma identifies what equivalence relations one gets as $E_{(k_n)}$ for various choices of (k_n). The lemma will be used in this section (only for a fixed parameter (k_n) as in (iv)) as well as in Section 5.

LEMMA 4.1. *Let L_k stand for $\{n: k = k_n\}$.*

(i) $E_{(k_n)} \cong 2^{\mathbb{N}}$ *iff for all but finitely many k, L_k is empty.*

(ii) $E_{(k_n)} \cong \mathbb{E}_0$ *iff for all k, L_k is finite.*

(iii) $E_{(k_n)} \cong \mathbb{E}_0 \times 2^{\mathbb{N}}$ *iff for all but finitely many k, L_k is finite, for infinitely many k, L_k is nonempty and, for some k, L_k is infinite.*

(iv) $E_{(k_n)} \cong \mathbb{E}_1$ *iff for infinitely many k, L_k is infinite.*

Proof. We start with an obvious claim.

Claim. $xE_{(k_n)}\, y$ precisely when for some k, $x(n) = y(n)$ whenever $k < k_n$.

The homeomorphism witnessing $\cong$ in points (i) and (ii) will be the identity.

(i) Let M be such that $L_k = \varnothing$ for all $k \geqslant M$. Note that $k_n < M$ for all n. It follows that for any $x, y \in 2^{\mathbb{N}}$, for any n, $x \,|\, nE_M\, y \,|\, n$, so $xE_{(k_n)}\, y$.

(ii) and (iii) Let M be large enough so that L_k is finite for all $k \geqslant M$. Let $X_0 = \{n: k_n < M\}$ and $X_1 = \{n: k_n \geqslant M\}$. Note that $\lim_n \{k_n: n \in X_1\} = \infty$. Thus, by the claim, $xE_{(k_n)}\, y$ precisely when $x(n) = y(n)$ for all but finitely many $n \in X_1$. In (ii), X_0 is finite, so this simply means that $x\mathbb{E}_0\, y$. In (iii), both X_0 and X_1 are infinite, so $xE_{(k_n)}\, y$ iff $x \,|\, X_1$ and $y \,|\, X_1$ are equal from some point on and $x \,|\, X_0$ and $y \,|\, X_0$ are arbitrary. So the following $f: 2^{\mathbb{N}} \to 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ witnesses $E_{(k_n)} \cong \mathbb{E}_0 \times 2^{\mathbb{N}}$: $f(x) = (y, z)$ with $y(m) = x(h_1(m))$ and $z(m) = x(h_0(m))$ where h_1 and h_0 are bijections between $\mathbb{N}$ and X_1 and X_0, respectively.

(iv) Let $a_m \in \mathbb{N}$, $m \in \mathbb{N}$, be such that $a_m < a_{m+1}$, $a_0 = 0$, and $\{n: a_m \leqslant k_n < a_{m+1}\}$ is infinite for each m. Put $X_m = \{n: a_m \leqslant k_n < a_{m+1}\}$. Let $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined by letting

$$h(m, p) = \text{ the } p\text{'th element of } X_m.$$

We start counting from 0, so $h(m,0) = \min X_m$. Let $f\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$ be defined by

$$f(x)(h(m,p)) = x(m,p).$$

Again by using the claim we see that f witnesses $\mathbb{E}_1 \cong E_{(k_n)}$.

In the following lemma, we grouped some easy but useful properties of the equivalence relations E_k.

LEMMA 4.2.

(i) *For each k, $E_k \subseteq E_{k+1}$, so if a is an E_k equivalence class and a' is an $E_{k'}$ equivalence class with $k \leqslant k'$, then $a \subseteq a'$, or $a \cap a' = \varnothing$.*

(ii) *For each equivalence class a of $E_k \mid 2^n$ either*

(a) *$b = \{s0, s1\colon s \in a\}$ is an $E_k \mid (2^{n+1})$ equivalence class or*

(b) *$b_0 = \{s0\colon s \in a\}$ and $b_1 = \{s1\colon s \subset a\}$ are two distinct $E_k \mid 2^{n+1}$ equivalence classes.*

If (a) holds in Lemma 4.2(ii), b will be called the extension of a; if (b) holds, b_0 is called the extension of a and b_1 the 1-extension of a.

THEOREM 4.3. *Let C be an indecomposable continuum. Assume that for some $\bar{x} \in C$ each proper subcontinuum containing $\bar{x}$ is hereditarily indecomposable. Then $\mathbb{E}_1 \sqsubseteq_c E_C$.*

Proof. Throughout this proof F_k, $k \in \mathbb{N}$, stand for the compact equivalence relations on C defined in Theorem 2.1. Now we construct a continuous embedding of $\mathbb{E}_1$ into E_C. We will use the representation of $\mathbb{E}_1$ defined in Lemma 4.1. For definiteness and ease of notation, we assume that (k_n) in Lemma 4.1 is chosen so that for any k, $k = k_n$ for infinitely many n, $k_0 = 0$, and $k_n \leqslant 1 + \max\{k_i\colon i < n\}$ for all n. Note that our choice of (k_n) implies that $k_n \leqslant n$.

Fix n. For each $t \in 2^n$, we will define $U_t \subseteq C$ nonempty open so that

(1) $\operatorname{diam}(U_t) \leqslant 1/(n+1)$, $t \in 2^n$;

(2) $\overline{U_{t0}}, \overline{U_{t1}} \subseteq U_t$, $t \in 2^{n-1}$;

(3) $t \neq s \Rightarrow U_t \cap U_s = \varnothing$, $t, s \in 2^n$;

(4) $\neg(tE_k s) \Rightarrow (U_t \times U_s) \cap F_k = \varnothing$, $t, s \in 2^n$.

Let $M = \max\{k_i\colon i \leqslant n\}$. For each $k \leqslant M$ and each equivalence class a of $E_k \mid 2^n$ we will define a proper subcontinuum $C_a \subseteq C$ so that

(5) if a is an $E_k \mid 2^n$ equivalence class and a' is an $E_{k'} \mid 2^n$ equivalence class, $k, k' \leqslant M$, then $C_a \subseteq C_{a'} \Leftrightarrow a \subseteq a'$ and $C_a \cap C_{a'} = \varnothing \Leftrightarrow a \cap a' = \varnothing$.

In the three conditions below, a is an $E_k \mid 2^n$ equivalence class for some $k \leqslant M$.

(6) $t \in 2^n$ and $t \in a$ imply $U_t \cap C_a \neq \varnothing$;

(7) if b is an $E_k \mid 2^{n-1}$ equivalence class and C_b is defined (that is, $k \leqslant \max\{k_i: i \leqslant n-1\}$), and a is its extension, then $C_a = C_b$;

(8) C_a is not included in an F_k equivalence class.

Note that (8) implies that C_a is not a one point set. Let $\bar{x} \in C$ be such that all proper subcontinua containing $\bar{x}$ are hereditarily indecomposable. Let $K_n = [\bar{x}]_{F_n}$. Then K_n is a continuum and $K_n \subseteq K_{n+1}$. By passing to a subsequence, we can assume additionally that $K_n \neq K_{n+1}$ and that K_0 contains more than one point. Let $d_n = \operatorname{dist}(K_n, C)$. Then $d_n \geqslant d_{n+1}$ and $d_n > 0$.

(9) for $k \leqslant \max\{k_i: i \leqslant n-1\}$, if b is an $E_k \mid 2^n$ equivalence class 1-extending a, an $E_k \mid (2^{n-1})$ equivalence class, then $\operatorname{dist}(C_a, C_b) \leqslant d_{n+1}/2^{n+1}$.

(10) $C_{2^n} = K_{M+1}$, here 2^n is the only $E_M \mid 2^n$ equivalence class.

Assume the construction has been carried out. Define $f: 2^{\mathbb{N}} \to C$ by $f(x) = \bigcap_n U_{x \mid n}$. By (1) and (2), f is well-defined and continuous, and by (3) it is 1-to-1. Let $x, y \in 2^{\mathbb{N}}$ be such that $\neg(x \mathbb{E}_1 y)$. Then $\forall k \, \exists n \, \neg(x \mid n E_k \, y \mid n)$ and it follows from (4) that $(f(x), f(y)) \notin \bigcup_k F_k = E_C$. Now let $x, y \in 2^{\mathbb{N}}$ be such that $x \mathbb{E}_1 y$, that is, $\exists k \, \forall n \; x \mid n E_k \, y \mid n$. Let a_n, $n \in \mathbb{N}$, be the equivalence class of $E_k \mid 2^n$ to which both $x \mid n$ and $y \mid n$ belong. Note that for each n, a_{n+1} is the extension or the 1-extension of a_n. For $n \geqslant \min\{i: k_i \geqslant k\} = n_0$, C_{a_n} are defined and, by (7) and (9), they form a Cauchy sequence. Let $\bar{C} = \lim_n C_{a_n}$. Again from (7) and (9) and the fact that $d_n \geqslant d_{n+1}$, taking into account $k \leqslant n_0$, we get

$$\operatorname{dist}(C_{a_{n_0}}, \bar{C}) \leqslant (1/2)\, d_{n_0+1} \leqslant (1/2)\, d_{k+1}.$$

By our choice of (k_n), $k = \max\{k_i: i \leqslant n_0\}$, so from (10), $C_{a_{n_0}} = K_{k+1}$, hence $\operatorname{dist}(K_{k+1}, \bar{C}) \leqslant (1/2)\, d_{k+1}$. It follows that

$$\begin{aligned}\operatorname{dist}(C, \bar{C}) &\geqslant \operatorname{dist}(C, K_{k+1}) - \operatorname{dist}(K_{k+1}, \bar{C}) \geqslant d_{k+1} - (1/2)\, d_{k+1} \\ &= (1/2)\, d_{k+1} > 0.\end{aligned}$$

So, $\bar{C}$ is proper. By (6) and (1), $f(x), f(y) \in \bar{C}$. Thus, $f(x)\, E_C\, f(y)$. (Recall that $\bar{C}$ is a continuum since limits of continua are continua.)

To carry out the construction, we will need the following technical claim.

Claim. *Let K be a hereditarily indecomposable continuum, and let $C^i \subseteq K$, $i \leqslant m \in \mathbb{N}$, be proper subcontinua. Then there exist continua $C_n^i \subseteq K$, $i \leqslant m$, $n \in \mathbb{N}$, such that*

(i) $\bigcup_{i \leqslant m} C_n^i \cap \bigcup_{i \leqslant m} C^i = \varnothing$ *for all n;*

(ii) $C_n^i \to C^i$ *as* $n \to \infty$ *for all* $i \leqslant m$;

(iii) $C_n^i \subseteq C_n^j \Leftrightarrow C^i \subseteq C^j$ *for all* $i, j \leqslant m$ *and all n.*

Proof of the claim. Recall that by [W, Theorem 3.1] each hereditarily indecomposable continuum K has property of Kelly: for any continuum $C \subseteq K$ and sequence of points $p_n \in K$, $n \in \mathbb{N}$, with $p_n \to p \in C$ there exists a sequence of continua (C_n) with $p_n \in C_n$ and $C_n \to C$.

SUBCLAIM. (i) *Let* $C' \subseteq K$ *be a proper subcontinuum. There exist* $C_n \subseteq K$, $n \in \mathbb{N}$, *such that for each n,* $C_n \cap C' = \varnothing$ *and* $C_n \to C'$ *as* $n \to \infty$.

(ii) *Let* $C'' \subseteq C' \subseteq K$ *be subcontinua. Let* C'_n, $n \in \mathbb{N}$, *be continua such that* $C'_n \to C'$. *Then there exist continua* $C''_n \subseteq C'_n$ *with* $C''_n \to C''$ *as* $n \to \infty$.

Proof of Subclaim. (i) Pick points $p_n \in K$, $n \in \mathbb{N}$, so that no p_n is chosen from the composant of K containing C' and $p_n \to p$, as $n \to \infty$ for some $p \in C'$. This is possible since each composant of K is meager in K. Now apply property of Kelly to this sequence (p_n) and to C'.

(ii) Pick $p_n \in C'_n$ with $p_n \to p$ for some $p \in C''$. Now apply property of Kelly to (p_n) and C'' to obtain a sequence (K_n) of continua. Since $p_n \in C'_n \cap K_n$ for each n, by hereditary indecomposability of K, we get that $K_n \subseteq C'_n$ or $C'_n \subseteq K_n$ for each n. Let $C''_n = K_n$ whenever the first possibility holds and $C''_n = C'_n$ whenever that latter holds. It is easy to check that this sequence works which finishes the proof of the subclaim.

Since K is hereditarily indecomposable, $C^i \subseteq C^j$, $C^j \subseteq C^i$, or $C^i \cap C^j = \varnothing$ for $i, j \leqslant m$. So we can find $i_0, \ldots, i_l$ such that C^{i_j}, $j < l$, are pairwise disjoint and each C^i is contained in some C^{i_j}. Using Subclaim (i), we can find $C_n^{i_j}$, $j < l$, $n \in \mathbb{N}$, such that $C_n^{i_j} \to C^{i_j}$, $C_n^{i_j} \cap \bigcup_{i \leqslant m} C^i = \varnothing$, and for each n the $C_n^{i_j}$ are pairwise disjoint. Fix j_0. Let C^{k_j}, $j < m = m(j_0)$, be pairwise disjoint chosen from the C^i's contained in $C^{i_{j_0}}$ so that each such C^i is included in some C^{k_j}. Using Subclaim (ii), we find $C_n^{k_j} \subseteq C_n^{i_{j_0}}$ pairwise disjoint continua with $C_n^{k_j} \to C^{k_j}$ as $n \to \infty$. We do it for each $j_0 \leqslant l$. We continue this process using repeatedly, though finitely many times, Subclaim (ii). This finishes the proof of the claim.

It remains to carry out the construction of the U_t's and the C_a's.

Step 0. Let $C_{2^0} = K_1$ and let $U_\varnothing$ be any open set with diameter not exceeding 1 and $U_\varnothing \cap K_1 \neq \varnothing$.

Step $n+1$. We will split the proof here into two cases. The proof in Case 2 consists actually of applying the argument of Case 1 inside a finite number of pairwise disjoint subcontinua of C. So, apart from giving half of the construction in Step $n+1$, Case 1 serves also as a lemma to Case 2.

Case 1: $k_{n+1} > \max\{k_i: i \leqslant n\}$. (Note that $k_{n+1} = 1 + \max\{k_i: i \leqslant n\}$.) Put $M = \max\{k_i: i \leqslant n\}$.

Let $C_{2^{n+1}} = K_{k_{n+1}+1}$. ($2^{n+1}$ is the only equivalence class of $E_{k_{n+1}} \mid 2^{n+1}$.) By the claim applied to $K = C_{2^{n+1}}$, we can find sequences of continua K_a^m, $m \in \mathbb{N}$, where a varies over the set of all $E_k \mid 2^n$ equivalence classes for $k \leqslant \max\{k_i: i \leqslant n\}$, such that

(a) $K_a^m \subseteq K_{k_{n+1}+1}$,

(b) $K_a^m \to C_a$ as $m \to \infty$,

(c) $K_a^m \subseteq K_{a'}^m \Leftrightarrow C_a \subseteq C_{a'}$,

(d) $K_{2^n}^m \cap C_{2^n} = \varnothing$. (Here 2^n is the only $E_M \mid 2^n$ equivalence class.)

Since, if a is an $E_k \mid 2^n$ equivalence class, $k \leqslant M$, $C_a = \lim_m K_a^m$, the following two points hold. First for m large enough K_a^m is not contained in an F_k equivalence class. If it were, so would be C_a as F_k is closed, and this would contradict our inductive assumption (8). Second, for $t \in a$ and for m large enough $K_a^m \cap U_t \neq \varnothing$. This is clear since U_t is open and $U_t \cap C_a \neq \varnothing$ by (6). Choose $m = m_0$ so that both these points hold for each $E_k \mid 2^n$ equivalence class a, $k \leqslant M$, and additionally $\operatorname{dist}(K_a^{m_0}, C_a) \leqslant d_{n+2}/2^{n+2}$. For a as above let $C_{b'} = K_a^{m_0}$ where b' is the 1-extension of a and let $C_b = C_a$ where b is the extension of a. By this point we have defined C_b for all $E_k \mid 2^{n+1}$ equivalence classes where $k \leqslant k_{n+1}$.

Note now that since $K_{2^n}^{m_0}$ or C_{2^n} is not contained in an F_M equivalence class, they must contain F_M equivalence classes of all its elements. (This is because $C_{2^{n+1}}$ is a hereditarily indecomposable and F_M equivalence classes are continua.) It follows that if $x \in K_{2^n}^{m_0}$ and $y \in C_{2^n}$, then $\neg(xF_M y)$. For $t \in 2^n$ and a an $\mathbb{E}_0 \mid 2^n$ equivalence class with $t \in a$, let $x_t \in K_a^{m_0} \cap U_t$ and $y_t \in C_a \cap U_t$. This is possible by the inductive assumption (6). Since F_M is closed, we can find V_t, W_t open with $x_t \in V_t$, $y_t \in W_t$ and so that $(V_s \times W_t) \cap F_M = \varnothing$ for $s, t \in 2^n$. Let U_{t0} be an open set with $y_t \in U_{t0}$, $\bar{U}_{t0} \subseteq W_t \cap U_t$ and U_{t1} an open set with $x_t \in U_{t1}$ and $\bar{U}_{t1} \subseteq V_t \cap U_t$. Obviously, we can make U_{t0} and U_{t1} be of arbitrarily small diameter.

The C_b's along with the U_t's are as required. Points (1–3) and (7–10) are evident from the construction. To check (4), let $s, t \in 2^{n+1}$ be such that $\neg(sE_k t)$ for some k. Obviously $k \leqslant M = \max\{k_i: i \leqslant n\}$ since all $s, t \in 2^{n+1}$ are $E_{k_{n+1}}$ related. If $\neg(s \mid nE_k t \mid n)$, then by the inductive assumption $(U_{s|n} \times U_{t|n}) \cap F_k = \varnothing$, so $(U_s \times U_t) \cap F_k = \varnothing$ as $U_s \subseteq U_{s|n}$ and $U_t \subseteq U_{t|n}$. Assume that $s \mid nE_k t \mid n$ and $\neg(sE_k t)$. Then by the construction we must have $s(n) = 0$ and $t(n) = 1$, or $s(n) = 1$ and $t(n) = 0$. Suppose the first is the case. Then $(U_s \times U_t) \cap F_K = \varnothing$ as $U_s \subseteq V_{s|n}$ and $U_t \subseteq W_{t|n}$. Since $k \leqslant M$ and so $F_k \subseteq F_M$, we have $(U_s \times U_t) \cap F_k = \varnothing$.

For (5), consider b, b' two equivalence classes of $E_k \mid 2^{n+1}$ and $E_{k'} \mid 2^{n+1}$, respectively. If k or k' is equal to k_{n+1}, say $k = k_{n+1}$, then $b = 2^{n+1}$ and so

$b' \subseteq b$. But also $C_b = K_{k_{n+1}+1} \supseteq C_{b'}$. So assume that $k, k' < k_{n+1}$. Then b is an extension or a 1-extension of some $E_k \mid 2^n$ equivalence class and similarly b' is an extension or a 1-extension of some $E_{k'} \mid 2^n$ equivalence class. The resulting four cases that need to be considered are similar to each other. We write out explicitly only the case when both of them are 1-extensions. Let a and a' be equivalence classes which are 1-extended by b and b', respectively. Then using the inductive assumption (5) as well as our choice of $K_a^{m_0}$ and $K_{a'}^{m_0}$, we have

$$b \subseteq b' \Leftrightarrow a \subseteq a' \Leftrightarrow C_a \subseteq C_{a'} \Leftrightarrow K_a^{m_0} \subseteq K_{a'}^{m_0} \Leftrightarrow C_b \subseteq C_{b'}.$$

Point (6) holds since $U_t \cap C_b \neq \varnothing$ for the $\mathbb{E}_0 \mid 2^{n+1}$ equivalence class b containing $t \in 2^{n+1}$, and by (5) above and $\mathbb{E}_0 \subseteq E_k$, $C_b \subseteq C_{b'}$ for each $E_k \mid 2^{n+1}$ equivalence class b' with $t \in b'$, $k \leqslant k_{n+1}$.

Case 2: $k_{n+1} \leqslant \max\{k_i : i \leqslant n\}$. Let b be an $E_k \mid 2^{n+1}$ equivalence class with $k_{n+1} \leqslant k \leqslant \max\{k_i : i \leqslant n\}$. Then b is the extension of an $E_k \mid 2^n$ equivalence class, say a. (Note that there are no 1-extensions in this case.) Let $C_b = C_a$.

Consider now all a which are $E_{k_{n+1}} \mid 2^n$ equivalence classes. Note that $C_{a_1} \cap C_{a_2} = \varnothing$ if a_1 and a_2 are such distinct equivalence classes. Fix C_a with a as above. We consider now all b which are $E_k \mid 2^{n+1}$ equivalence classes with $k < k_{n+1}$ and which have the following property: b extends or 1-extends an $E_k \mid 2^n$ equivalence class included in a. We define C_b for such b's and U_t with $t \in b$ by applying the claim with $K = C_a$ in a manner similar to that in Case 1. (If $k_{n+1} = 0$, one has to only define U_t for $t \in 2^{n+1}$ and does not need to use the claim at all. This is done be defining U_{s0}, U_{s1}, $s \in 2^n$, to be two open disjoint sets with $\bar{U}_{s0}, \bar{U}_{s1} \subseteq U_s$ and such that U_{s0}, U_{s1} and U_s intersect exactly the same C_a's.) This finishes the construction.

Corollary 4.4. *Assume C is an indecomposable continuum with $\bar{x} \in C$ such that each proper subcontinuum containing $\bar{x}$ is hereditarily indecomposable. Then $E_C \approx_B \mathbb{E}_1$.*

Proof. Immediate from Theorem 4.3, Corollary 2.2, and the classification of hyperfinite equivalence relations [KL] (see the introduction 1.2).

5. THE CONDITIONS $E_C \approx_B \mathbb{E}_1$ AND $E_C \approx_B \mathbb{E}_0$

We will be mostly concerned with the condition $E_C \approx_B \mathbb{E}_1$. But all our results easily translate to results about $E_C \approx_B \mathbb{E}_0$ since the class of indecomposable continua is simple ($\mathbf{\Pi}_2^0$, see below) and the two conditions are complements of each other among indecomposable continua.

We show that the family

$$\{C \subseteq [0,1]^{\mathbb{N}} : C \text{ an indecomposable continuum and } E_C \approx_B \mathbb{E}_1\}$$

is $\mathbf{\Sigma}^1_1$. Curiously, the proof of this upper estimate on the complexity uses effective descriptive set theory. Note that the obvious estimate for $E_C \approx_B \mathbb{E}_1$, which by [KL, Theorem 2.1] is equivalent to $\mathbb{E}_1 \sqsubseteq_c E_C$, is

$$\exists f\colon (2^{\mathbb{N}})^{\mathbb{N}} \to [0,1]^{\mathbb{N}} \text{ continuous } \forall x, y \in (2^{\mathbb{N}})^{\mathbb{N}}\ (x \mathbb{E}_1 y \Leftrightarrow f(x)\, E_C\, f(y))$$

and gives only that the condition is $\mathbf{\Sigma}^1_2$. Similarly, a different estimate for its complement, based on Corollary 3.3(ii),

$$\exists f\colon C \to 2^{\mathbb{N}} \text{ Borel } \forall x, y \in C\, (x E_C y \Leftrightarrow f(x)\, \mathbb{E}_0\, f(y)) \tag{2}$$

can be seen to be only $\mathbf{\Sigma}^1_2$, so gives $\mathbf{\Pi}^1_2$ for $E_C \approx_B \mathbb{E}_1$. However, this last approach can be refined using effective descriptive set theory to give $\mathbf{\Pi}^1_1$ for the complement of $E_C \approx_B \mathbb{E}_1$ and so the desired $\mathbf{\Sigma}^1_1$ for $E_C \approx_B \mathbb{E}_1$ itself. I tried to provide enough details in the proof here so that even the reader not familiar with the recursive aspects of descriptive set theory should be able to follow the argument after consulting [Mo] or [HKL, Section 3].

We think of $[0,1]^{\mathbb{N}}$, $2^{\mathbb{N}}$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{N}$ and their products as being recursively presented in the sense of [Mo, 3B]. Recursive presentations of all these spaces, except for $[0,1]^{\mathbb{N}}$, are done explicitly in [Mo]. It is not difficult to carry it out for $[0,1]^{\mathbb{N}}$, as well (see [Mo, 3B.6]). This, in particular, means that we can talk about $\Sigma^0_1(r)$, $\Pi^0_1(r)$, $\Delta^1_1(r)$, $\Sigma^1_1(r)$, and $\Pi^1_1(r)$ subsets of these spaces for parameters $r \in \mathbb{N}^{\mathbb{N}}$. (In these symbols Σ and Π are lightface which makes a difference see [Mo] or [HKL].) We have a metric d on $[0,1]^{\mathbb{N}}$ for which the set $\{(k, n, x, y) \in \mathbb{N} \times \mathbb{N} \times [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} : d(x, y) < k/(n+1)\}$ is Σ^0_1. And similarly with the condition $d(x, y) < k/(n+1)$ replaced by $d(x, y) > k/(n+1)$. We also have a topological basis $\{U_n : n \in \mathbb{N}\}$ of $[0,1]^{\mathbb{N}}$ for which $\{(n, x) \in \mathbb{N} \times [0,1]^{\mathbb{N}} : x \in U_n\}$ is Σ^0_1. We think of $2^{\mathbb{N}}$ as naturally embedded in $\mathbb{N}^{\mathbb{N}}$.

We will need a coding for $\mathbf{\Delta}^1_1$ subsets of $X = [0,1]^{\mathbb{N}} \times 2^{\mathbb{N}}$. (See [HKL, 3.3, 3.4].) There exists a Π^1_1 set $D \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ and sets $W^{\Sigma}, W^{\Pi} \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times X$ such that $W^{\Sigma} \in \Sigma^1_1$, $W^{\Pi} \in \Pi^1_1$, for each $(x, n) \in D$, $W^{\Sigma}_{(x,n)} = W^{\Pi}_{(x,n)}$, and $B \subseteq X$ is $\Delta^1_1(r)$ for $r \in \mathbb{N}^{\mathbb{N}}$ iff $B = W^{\Sigma}_{(r,n)} = W^{\Pi}_{(r,n)}$ for some $n \in \mathbb{N}$.

THEOREM 5.1. *The family of all indecomposable subcontinua C of $[0,1]^{\mathbb{N}}$ for which $E_C \approx_B \mathbb{E}_1$ is $\mathbf{\Sigma}^1_1$.*

Proof. Let K be a compact subset of $[0,1]^{\mathbb{N}}$. Let $r \in 2^{\mathbb{N}}$ be called a code of K if $[0,1]^{\mathbb{N}} \setminus K = \bigcup \{U_n : r(n) = 1\}$. In such a situation, we write $K = K_r$. Note that the condition $x \notin K_r$ is Δ^1_1 on (x, r) since it is equivalent to

$\exists j\, r(j)=1$ and $x \in U_j$. Therefore, so is the condition $x \in K_r$. Define a relation E_K between points in K: $xE_K y$ holds when x and y lie in a continuum contained in K but not equal to K. So, if K is an indecomposable continuum, E_K is the composant equivalence relation. In general, however E_K is not an equivalence relation.

The idea of the proof is as follows. We check that if C is an indecomposable continuum with a code r, then E_C can be represented as the union of an increasing sequence (F_n) of equivalence relations with this sequence in $\Pi^0_1(w)$ for some $w \in \Delta^1_1(r)$, $w \in \mathbb{N}^{\mathbb{N}}$. From this, using the effective version of the Kechris–Louveau dichotomy, we get that if $E_C \leqslant_B \mathbb{E}_0$, then one can find f which is in $\Delta^1_1(r)$ and which witnesses it. This allows us to replace the quantifier $\exists f$ in the condition $E_C \leqslant_B \mathbb{E}_0$ as written out in (2) by $\exists f \in \Delta^1_1(r)$. This change makes the condition Π^1_1. Now for the details.

CLAIM 1. *Let r be a code for K. E_K is $\Delta^1_1(r)$.*

Proof of Claim 1. For $x, y \in K$, $xE_K y$ if, and only if, x and y lie in the same component of a proper compact subset L of K. Two points x, y of L lie in the same component of L precisely when for each $\varepsilon > 0$, there exists a finite sequence of points $x_0, \ldots, x_m \in L$ such that $d(x, x_0) < \varepsilon$, $d(y, x_m) < \varepsilon$, $d(x_i, x_{i+1}) < \varepsilon$ for $i < m$. It follows therefore that $xE_K y$ is equivalent to

(3)

$$\exists n\, \forall k\, \exists m\, \exists x_0, \ldots, x_m\, \exists z\, (z \in K, \forall i \leqslant n\, (x_i \in K \text{ and } d(z, x_i) > 1/(n+1)), \\ d(x, x_0) < 1/(k+1), d(y, x_m) < 1/(k+1), \forall i < n\, d(x_i, x_{i+1}) < 1/(k+1)).$$

This shows that E_K is $\Sigma^1_1(r)$. Note that the conditions on the x_i's and z following the quantifiers $\exists x_0, \ldots, x_m$ and $\exists z$ are relatively open in K. Since points which are $\Delta^1_1(r)$ are dense in K by [Mo, 4F.11] as K is compact, we can replace these quantifiers by $\exists x_0 \in \Delta^1_1(r), \ldots, x_m \in \Delta^1_1(r)$ and $\exists z \in \Delta^1_1(r)$ without changing the meaning of the condition. After this modification the condition becomes $\Pi^1_1(r)$ by [Mo, 4D.3], and the claim is proved.

CLAIM 2. *Let C be an indecomposable subcontinuum of $[0, 1]^{\mathbb{N}}$ with a code r. Then there exists a set $F \subseteq \mathbb{N} \times [0, 1]^{\mathbb{N}} \times [0, 1]^{\mathbb{N}}$ such that*

(i) *F is $\Pi^0_1(w)$ for some $w \in \mathbb{N}^{\mathbb{N}}$ with $w \in \Delta^1_1(r)$;*

(ii) *each F_n is an equivalence relation on C;*

(iii) *$F_n \subseteq F_{n+1}$;*

(iv) *$\bigcup_n F_n = E_C$.*

Proof of Claim 2. Let $z_0 \in C$ be $\Delta_1^1(r)$. Such a z_0 exists by [Mo, 4F.11]. By Claim 1, $\{x \in C : xE_C z_0\}$ is $\Delta_1^1(r)$. Therefore, its complement in C is $\Delta_1^1(r)$ and it is comeager in C by Kuratowski's theorem that each composant is meager [N, 11.14 and 6.19]. By [Mo, 4F.20], this allows us to pick z_1 in this complement with z_1 in $\Delta_1^1(r)$. (Note that there is a misprint in the statement of [Mo, 4F.20]: $\Sigma_1^1(z)$ there should read $\Pi_1^1(z)$.) This z_1 does not belong to the composant of z_0. Now define $F \subseteq \mathbb{N} \times C \times C$ as follows

$$(n, x, y) \in F \Leftrightarrow x = y \text{ or } \forall k\ \exists m, p \exists x_0, \ldots, x_m\ \exists w_0, \ldots, w_p$$

(a) $x_0, \ldots, x_m, w_0, \ldots w_p \in C$,

(b) $d(x, x_0) < 1/(k+1), d(y, x_m) < 1/(k+1)$,
$\forall i < m\, d(x_i, x_{i+1}) < 1/(k+1)$,

(c) $d(w_0, z_0) < 1/(k+1), \forall i < p\, d(w_i, w_{i+1}) < 1/(k+1)$,

(d) $\forall j \leqslant p\, d(y_j, z_1) > 1/(n+1) - 1/(k+1)$

(e) $\forall i \leqslant m\, (d(x_i, z_0) > 1/(n+1) - 1/(k+1)$ or
$\exists j \leqslant p\, d(x_i, y_j) < 1/(k+1))$.

This clearly gives a $\Sigma_1^1(r)$ definition of F. The conditions following the quantifiers $\exists x_0, \ldots, x_m$ and $\exists y_0, \ldots, y_p$ are relatively open in C, so F is $\Pi_1^1(r)$ by an argument as in the proof of Claim 1. Thus, $F \in \Delta_1^1(r)$.

For each n, $(n, x, y) \in F$ precisely when (1) from the proof of Theorem 2.1 holds. We present an argument for this maintaining notation from Theorem 2.1. Assume (1) holds for $(x, y) \in C \times C$. If $x = y$, then $(n, x, y) \in F$. Otherwise, there exists a continuum $L \subseteq C$ such that $x, y \in L \subseteq (C \setminus V_n) \cup C_n$. Fix k and put $\varepsilon = 1/(k+1)$. Let $x_0, \ldots, x_m \in L$, $w_0, \ldots, w_p \in C_n$ be ε-nets for L and C_n, respectively, numbered so that (b) and (c) hold. Now (a) is obvious and (d) holds since each y_j is in C_n. To check (e), fix $i \leqslant m$ and assume that $d(x_i, z_0) \leqslant 1/(n+1) - 1/(k+1)$. This means that $x_i \in V_n$. Since also $x_i \in L$, we have $x_i \in C_n$. Thus, since the y_j's constitute an ε-net, $d(x_i, y_j) < \varepsilon$ for some j. To see the other direction, assume $(n, x, y) \in F$. If $x = y$, then (1) holds for (x, y). If not, for each k, fix $x_0^k, \ldots, x_{m_k}^k$ and $w_0^k, \ldots, w_{p_k}^k$ for which (a)–(e) hold. Pick a sequence (k_l) so that $\{x_0^{k_l}, \ldots, x_{m_{k_l}}^{k_l}\} \to L$, $\{w_0^{k_l}, \ldots, w_{m_{k_l}}^{k_l}\} \to K$ for some $L, K \in \mathscr{K}(C)$. By (b), L is a continuum containing x and y. Let $w \in L \cap V_n$. We need to see that $w \in C_n$. By (e), $w \in K$. By (c), K is a continuum containing z_1 and, by (d), it is contained in $C \setminus \{x \in C : d(x, z_1) < 1/(n+1)\}$. Since C_n is the component of this last set containing z_1, we have $K \subseteq C_n$, so $w \in C_n$, and we are done.

By the properties of the F_n's from (1), we get (ii), (iii), (iv), and also that F is in $\mathbf{\Pi}_1^0$. Since $F \in \Delta_1^1(r)$, by (the simplest case of) Louveau's theorem [Mo, 4F.14], [Lo, Theorem A], we can find w as in (i).

CLAIM 3. *Let C be an indecomposable subcontinuum of $[0,1]^{\mathbb{N}}$ with a code r. If $\mathbb{E}_1 \not\leqslant_B E_C$, then there exists $f: C \to 2^{\mathbb{N}}$ in $\Delta^1_1(r)$ which witnesses $E_C \leqslant_B E_0$.*

Proof of Claim 3. Let w be as in Claim 2. By [KL, Theorem 2.1], it follows from Claim 2, that if $\mathbb{E}_1 \not\leqslant_B E_C$, then there exists $f: C \to 2^{\mathbb{N}}$ in $\Delta^1_1(w)$ which witnesses $E_C \leqslant_B E_0$. But since $w \in \Delta^1_1(r)$, f is in $\Delta^1_1(r)$, which proves the claim.

Recall the coding for $\mathbf{\Delta}^1_1$ subsets of $[0,1]^{\mathbb{N}} \times 2^{\mathbb{N}}$ as explained in the paragraph preceding Theorem 5.1. Define a set A of all codes $r \in 2^{\mathbb{N}}$ which satisfy the following formula

$$\exists n\,(r,n) \in D,\ f = W^{\Pi}_{(r,n)} \text{ is a function with domain } K \text{ and}$$

$$\forall x, y(xE_K y \Leftrightarrow f(x)\, \mathbb{E}_0\, f(y)).$$

Note that, using Claim 1, this formula will show that A is Π^1_1, if we can prove that the condition "$(r,n) \in D, W^{\Pi}_{(r,n)}$ is a function with domain K" is Π^1_1 on (r,n). But this is clear since it is the conjunction of the following four formulas

1. $(r,n) \in D$;
2. $\forall x\, \forall y \in 2^{\mathbb{N}}\, (x \notin K \Rightarrow (r,n,x,y) \notin W^{\Sigma})$;
3. $\forall x\, \forall y_1, y_2 \in 2^{\mathbb{N}}\, (x \in K, (r,n,x,y_1), (r,n,x,y_2) \in W^{\Sigma} \Rightarrow y_1 = y_2)$;
4. $\forall x\, (x \in K \Rightarrow \exists y \in 2^{\mathbb{N}}, y \in \Delta^1_1(r,x)\, (r,n,x,y) \in W^{\Pi})$.

The formula in 4 is Π^1_1 by [Mo, 4D.3] and it holds if $W^{\Pi}_{(r,n)}$ is a function on K by [Mo, 4C.3]. Thus, A is Π^1_1.

Let $\phi: \mathscr{K}([0,1]^{\mathbb{N}}) \to 2^{\mathbb{N}}$ be defined by $\phi(K) = \{n: U_n \cap K = \varnothing\}$. Note that ϕ is Borel and $\phi(K)$ is a code for K. It follows now from Claim 3 that the set of all indecomposable continua C with $\mathbb{E}_1 \not\leqslant_B E_C$ is equal to

$$\phi^{-1}(A) \cap \{K \in \mathscr{K}([0,1]^{\mathbb{N}}) : K \text{ is an indecomposable continuum}\}$$

Since the set of all indecomposable continua is $\mathbf{\Pi}^0_2$, this shows that the set of continua C with $\mathbb{E}_1 \not\leqslant_B E_C$ is $\mathbf{\Pi}^1_1$, and the theorem follows.

In the next theorem, we construct a class of examples of indecomposable subcontinua of $[0,1]^{\mathbb{N}}$. It will be used to show that the condition $E_C \approx_B \mathbb{E}_1$ is a complicated one which will be manifested by the fact that the family of all indecomposable subcontinua C of $[0,1]^{\mathbb{N}}$ for which $E_C \approx_B \mathbb{E}_1$ is not Borel, in fact, it is $\mathbf{\Sigma}^1_1$-hard. This together with the upper estimate from Theorem 5.1 gives a precise location of the descriptive set theoretic complexity of this family of continua—it is $\mathbf{\Sigma}^1_1$-complete. It may be worth comparing this with the complexity of the family of all continua $\mathscr{C}$, all indecomposable continua $\mathscr{I}$, and all hereditarily indecomposable continua $\mathscr{H}$. These are all known to be low. $\mathscr{C}$ is a closed subset of $\mathscr{K}([0,1]^{\mathbb{N}})$, see [N, 4.17]. $\mathscr{I}$ is a G_δ, that is, $\mathbf{\Pi}_2^0$, since

$$K \in \mathscr{I} \Leftrightarrow \forall L_1, L_2 \in \mathscr{C}\,(L_1 \cup L_2 = K \Rightarrow L_1 = K \text{ or } L_2 = K)$$

and the right side is a coprojection of a G_δ set along a compact axis. Similarly $\mathscr{H}$ is a G_δ since

$$K \in \mathscr{H} \Leftrightarrow \forall L \in \mathscr{C}\,(L \subseteq K \Rightarrow L \in \mathscr{I})$$

and the right side is again a coprojection of a G_δ set along a compact axis. Thus, the conditions $E_C \approx_B \mathbb{E}_1$ and $E_C \approx_B \mathbb{E}_0$ are much more complicated than being in $\mathscr{C}$, $\mathscr{I}$, or $\mathscr{H}$. It may be interesting in this context to mention that the family of all hereditarily *decomposable* continua is $\mathbf{\Pi}_1^1$-complete by a result of Darji [D].

We consider the space $2^{2^{<\mathbb{N}}}$, that is, the product of countably many copies of $2 = \{0, 1\}$ with the copies indexed by elements of $2^{<\mathbb{N}}$. With its product topology this is a compact metric space, in fact, homeomorphic to the Cantor set. Let

$$\mathrm{IF'} = \{z \in 2^{2^{<\mathbb{N}}} : \exists \alpha \in 2^{\mathbb{N}}\, \exists^\infty n\, \alpha \,|\, n \in z\}.$$

It is well known, and follows directly from [K, Theorem 27.1], that IF' is a $\mathbf{\Sigma}_1^1$-complete set. Put $\mathrm{WF'} = 2^{2^{<\mathbb{N}}} \setminus \mathrm{IF'}$. (WF' stands for wellfounded and IF' for illfounded.)

THEOREM 5.2. *There exists a continuous function* $2^{2^{<\mathbb{N}}} \ni z \to C^z \in \mathscr{K}([0,1]^{\mathbb{N}})$ *such that*

(i) *C^z is an indecomposable continuum for each z;*
(ii) *if $z \in IF'$, then $\mathbb{E}_1 \sqsubseteq_c E_{C^z}$;*
(iii) *if $z \in WF'$, then $E_{C^z} \leqslant_B \mathbb{E}_0$.*

In particular, the family of all indecomposable subcontinua C of the Hilbert cube for which $E_C \approx_B \mathbb{E}_1$ is $\mathbf{\Sigma}_1^1$-hard, whence it is not Borel.

Proof. By an interval of a linear order we mean a connected set. If A is a subset of the domain of a linear order, let $[A]$ denote the smallest interval containing A. We will write $[a, b]$ for $[\{a, b\}]$ and $[a, b, c]$ for $[\{a, b, c\}]$. (So, even if $b \leqslant a$, $[a, b]$ stands for the closed interval with endpoints in a and b.) If $<$ is a linear order and A, B are subsets of its domain, we write $A < B$ if each element of A precedes in $<$ all elements of B. If x is in the domain of $<$, we write $A < x$ to indicate $A < \{x\}$ and $x < A$ for $\{x\} < A$. Let $(I, <_1)$ and $(J, <_2)$ be two linear orders. The *lexicographic order* $<_l$ on $I \times J$ is defined by letting $(x_1, y_1) <_l (x_2, y_2)$ if $x_1 <_1 x_2$, or $x_1 = x_2$ and $y_1 <_2 y_2$. The *antilexicographic order* $<_a$ on is defined by letting $(x_1, y_1) <_a (x_2, y_2)$ if $y_1 <_2 y_2$, or $y_1 = y_2$ and $x_1 <_1 x_2$. The set $2 = \{0, 1\}$ is always ordered so that 0 precedes 1 and the unit interval $[0, 1]$ is always equipped with its natural order.

The proof will be split into four parts. Most of the construction is essentially combinatorial in nature and consists of producing sequences of finite linear orders with maps between them. These sequences of finite linear orders are used, on the one hand, to represent $\mathbb{E}_1$ and, on the other hand, to define, what might be called, "discrete continua." These discrete continua are turned into continua by viewing the finite linear orders as suborders of $[0, 1]$ with the natural order, extending the maps between them to continuous maps between two copies of $[0, 1]$, and taking inverse limits. The procedure of embedding $\mathbb{E}_1$ exploits the connection between the two usages of sequences of linear orders: in representing $\mathbb{E}_1$ and in defining continua. The technique of reducing the composant equivalence relation to $\mathbb{E}_0$ is somewhat complicated with complications arising mainly from the process of turning discrete continua into continua.

Part 1: *A Representation of* $E_{(k_n)}$.

We will need to define a doubling operation on finite linear orders. Let $(I, <)$ be a finite linear order and let $I_0, I_1, \ldots, I_n$ be pairwise disjoint subintervals of I. Now $D(<; (I_0, \ldots, I_n))$ is described as follows. Its domain is

$$J = (I \times \{0\}) \cup \left(\bigcup_{k \leqslant n} I_k \times \{1\} \right).$$

The order $<'$ on J is defined by the following conditions. Let $(x, i), (y, j) \in J$.

(i) if x, y do not both belong to the same interval I_k for some $k \leqslant n$, let $(x, i) <' (y, j)$ precisely when $x < y$;

(ii) $<' \mid I_k \times 2$, for $k \leqslant n$, is the antilexicographic order where I_k is taken with $< \mid I_k$.

There is a natural projection $p\colon J \to I$ defined by letting $p(x, i) = x$ if $x \in I$ and $i \in 2$. We also define the doubling operation D in the case when there are no intervals I_k by letting $D(<; \varnothing)$ be $(I \times \{0\}, <')$ with $<'$ defined as in (i) above.

There are finitely many linear orders of 2^n. Let $\mathscr{L}_n$ be the set of all such linear orders. Let T be the set of all sequences of natural numbers (k_n) such that $k_0 = 0$, $1 \leqslant k_{n+1} \leqslant 1 + \max\{k_i : i \leqslant n\}$, and (k_n) is unbounded. Let $M_n = \max\{k_i : i \leqslant n\}$. Put $N_k = \min\{n : M_n = k\} = \min\{n : k_n = k\}$. We will associate with each $(k_n) \in T$ a sequence $(<^n) \in \prod_n \mathscr{L}_n$. So fix $(k_n) \in T$. Define a sequence of finite sequences of natural numbers

$$0 = i_0^n < i_1^n < \cdots < i_{M_n}^n = n$$

as follows:

$i_0^0 = 0;$

$i_k^{n+1} = i_k^n$ if $k < k_{n+1}$;

$i_k^{n+1} = i_k^n + 1$ if $k_{n+1} \leqslant k \leqslant M_n$;

$i_{k_{n+1}}^{n+1} = n + 1$ if $k_{n+1} = M_n + 1$.

Now define a sequence of linear orders $<^n$. The order $<^n$ is defined on 2^n. A subinterval of 2^n with respect to $<^n$ with 2^p elements will be called a *subinterval of rank p*. Let $<^0$ be the only possible order on the one element set 2^0. Assume $<^n$ has been defined. Split 2^n into $<^n$-intervals of rank $i_{k_{n+1}-1}^{n+1}$: $I_0, I_1, \ldots, I_L$. Here $L = 2^{n - i_{k_{n+1}-1}^{n+1}} - 1$. Define

$$(2^{n+1}, <^{n+1}) = D(<^n; (I_0, \ldots, I_L)).$$

Note that since $\bigcup_{i \leqslant L} I_i = 2^n$, the domain of $D(<^n; (I_0, \ldots, I_L))$ is indeed 2^{n+1}. Note also that the natural projection $p : 2^{n+1} \to 2^n$ is in this case simply $2^{n+1} \notin s \to s \mid n \in 2^n$. Define π^n, $n \geqslant 1$, mapping the set of all subintervals of 2^n with respect to $<^n$ to the set of subintervals of 2^{n-1} with respect to $<^{n-1}$. If I is a subinterval of 2^n, let $\pi^n(I)$ be $[\{s \mid (n-1) : s \in I\}]$. Let also π_k^m for $0 \leqslant k < m$ be $\pi^m \circ \cdots \circ \pi^{k+1}$, and let π_n^n be the identity function. We call π_k^m, $m \geqslant k$, the sequence of interval maps associated with $(<^n)$.

We need one more definition involving the linear orders $<^n$. Split 2^n into subintervals of equal length. Each of these subintervals will be called a *regular subinterval of* 2^n.

The proof of Lemma 5.3 is written up in greater detail than the rest of the proof of Theorem 5.2. To show Theorem 5.2, we will have to check quite a few statements by induction. In the proof of Lemma 5.3, we spell out in detail such inductive arguments; in the remainder of the proof, we leave it to the reader to do that.

LEMMA 5.3. *Let $(<^n)$ be associated with $(k_n) \in T$ as above, and let π_k^m, $m \geqslant k$, be the sequence of interval maps associated with $(<^n)$. Then for $x, y \in 2^{\mathbb{N}}$, the following conditions are equivalent*

(i) $xE_{(k_n)}\,y$;

(ii) $\exists k\,\forall n \geqslant N_k\; x\,|\,n$ *and* $y\,|\,n$ *belong to a regular subinterval of* 2^n *of rank* i^n_k;

(iii) $\exists k\,\forall m \geqslant k\,\pi^m_k([x\,|\,m, y\,|\,m]) \neq 2^k$.

Proof. We leave checking the following fact to the reader.

CLAIM 1. *Let* $s, t \in 2^{n+1}$.

(a) *If* $s(n) \neq t(n)$, *then* s *and* t *do not belong to a regular subinterval of rank* $p \leqslant i^{n+1}_{k_{n+1}-1}$.

(b) *If* $s(n) = t(n)$, *then, for* $p \leqslant i^{n+1}_{k_{n+1}-1}$, s *and* t *belong to a regular subinterval of rank* p *iff* $s\,|\,n$ *and* $t\,|\,n$ *belong to such an interval.*

(c) *For* $p > i^{n+1}_{k_{n+1}-1}$, s *and* t *belong to a regular subinterval of rank* p *iff* $s\,|\,n$ *and* $t\,|\,n$ *belong to a regular subinterval of rank* $p-1$.

We are now fixing k *till the end of proof of Claim* 6.

CLAIM 2. *For all* $n \geqslant N_k$, E_k *equivalence classes on* 2^n *are regular intervals of rank* i^n_k.

Proof of Claim 2. This is proved by induction on n. If $n = N_k$, by definition of E_k, 2^n is its only equivalence class on 2^n. Also $i^n_k = n$, so we get the conclusion. Assume the statement is true for n. Let $s, t \in 2^{n+1}$.

Case 1: $k < k_{n+1}$. Assume sE_kt. Then $s(n) = t(n)$ and $s\,|\,nE_kt\,|\,n$. By our inductive assumption, $s\,|\,n$ and $t\,|\,n$ lie in some regular subinterval I of rank i^n_k. Note also that, since $k < k_{n+1}$, $i^n_k = i^{n+1}_k \leqslant i^{n+1}_{k_{n+1}-1}$. Then by Claim 1b, s and t lie in a regular subinterval of rank $i^n_k = i^{n+1}_k$.

Assume $\neg(sE_kt)$. Then either $s(n) \neq t(n)$ or $\neg(s\,|\,nE_kt\,|\,n)$. Suppose first that $s(n) \neq t(n)$. Then by Claim 1a, s and t do not lie in the same regular interval of rank p with $p \leqslant i^{n+1}_{k_{n+1}-1}$, and since $k < k_{n+1}$, $i^{n+1}_k \leqslant i^{n+1}_{k_{n+1}-1}$. Now assume $\neg(s\,|\,nE_kt\,|\,n)$ and $s(n) = t(n)$. Then by our inductive assumption $s\,|\,n$ and $t\,|\,n$ do not belong to the same interval of rank i^n_k. So since $i^n_k = i^{n+1}_k \leqslant i^{n+1}_{k_{n+1}-1}$, by Claim 1b, s and t do not belong to a regular interval of rank $i^n_k = i^{n+1}_k$, we are done.

Case 2: $k \geqslant k_{n+1}$. Assume sE_kt, then $s\,|\,nE_kt\,|\,n$. Thus, by inductive assumption, $s\,|\,n$ and $t\,|\,n$ are included in a regular subinterval of 2^n of rank i^n_k. Note that $i^{n+1}_k = i^n_k + 1$ or $i^{n+1}_k = n+1$. In the latter case, clearly s and t lie in the same interval of rank i^{n+1}_k. Otherwise, $i^{n+1}_k > i^{n+1}_{k_{n+1}-1}$ and $s\,|\,n$ and $t\,|\,n$ belong to the same regular subinterval of rank $i^n_k = i^{n+1}_k - 1$, so by Claim 1c, s and t belong to the same regular subinterval of rank i^{n+1}_k as required.

Now assume $\neg(sE_kt)$. Then $\neg(s|nE_kt|n)$. If s and t were included in a regular interval of rank i_k^{n+1}, then, since $i_k^{n+1} \geqslant i_{k_{n+1}}^{n+1} > i_{k_{n+1}-1}^{n+1}$, by Claim 1c $s|n$ and $t|n$ would be in a regular interval of rank $i_k^{n+1}-1 = i_k^n$. But this would mean, by our inductive assumption that $s|nE_kt|n$, contradiction proving Claim 2.

For $n \geqslant n_k$ let $I_0^n, I_1^n, \cdots, I_{L_n}^n$ list all regular subintervals of 2^n of rank i_k^n. So $L_n = 2^{n-i_k^n}-1$. We additionally assume that each element in I_i^n is smaller than any element of I_j^n if $i<j$.

Claim 3. *$\pi^{n+1}(I_j^{n+1}) = I_i^n$ for some $i \leqslant L_n$ if $n \geqslant N_k$ and $j \leqslant L_{n+1}$. Moreover,* $(\min I_j^{n+1})|n = \min I_i^n$ *and* $(\max I_j^{n+1})|n = \max I_i^n$.

Proof of Claim 3. Fix I_j^{n+1} with $j \leqslant L_{n+1}$. It suffices to show that for some $i \leqslant L_n, \pi^{n+1}(I_j^{n+1}) \subseteq I_i^n, (\min I_j^n)|n = \min I_i^n$ and $(\max I_j^n)|n = \max I_i^n$. As in the proof of Claim 2, we consider two cases.

Case 1: $k < k_{n+1}$. By Claim 1a, for any $s_1, s_2 \in I_j^{n+1}$, $s_1(n) = s_2(n)$. Note that we have $i_k^n = i_k^{n+1} \leqslant i_{k_{n+1}-1}^{n+1}$; thus, by Claim 1b, the set $\{s|n : s \in I_j^{n+1}\}$ is included in a regular subinterval of rank i_k^n. It follows that $\pi^{n+1}(I_j^{n+1}) \subseteq I_i^n$ for some i. Moreover, in the situation we are considering, by definition of $<^{n+1}$, the mapping $I_j^{n+1} \notin s \to s|n \in I_i^n$ is increasing between $<^{n+1}$ and $<^n$, and since I_j^{n+1} and I_i^n have the same number of elements, as $i_k^{n+1} = i_k^n$, we actually have $(\min I_j^{n+1})|n = \min I_i^n$ and the same for max.

Case 2: $k \geqslant k_{n+1}$. If $k = k_{n+1} = M_n+1$, $L_{n+1} = 0$ and $I_0^{n+1} = 2^{n+1}$ and the conclusion is obvious. If $k \leqslant M_n$, then $i_k^n + 1 = i_k^{n+1} > i_{k_{n+1}-1}^{n+1}$, and it follows directly from Claim 1c that $\pi^{n+1}(I_j^{n+1}) = I_i^n$ for some i. There are $2^{i_k^{n+1}} = 2 \cdot 2^{i_k^n}$ elements in I_j^{n+1}. By definition of $<^{n+1}$ the mapping $s \to s|n$ is increasing on each of the sets

$$\{s \in I_j^{n+1} : s(n) = 0\} \text{ and } \{s \in I_j^{n+1} : s(n) = 1\},$$

the elements of the first of these sets precede in $<^{n+1}$ the elements of the second one, and both these sets have $2^{i_k^n}$ elements. Therefore,

$$\min I_i^n = (\min\{s \in I_j^{n+1} : s(n) = 0\})|n = (\min I_j^{n+1})|n,$$

and similarly for max. Thus Claim 3 is proved.

Claim 4. *If $n \geqslant m \geqslant N_k$ and $j \leqslant L_n$, then $\pi_m^n(I_j^n) = I_i^m$ for some $i \leqslant L_m$.*

Proof of Claim 4. This claim follows immediately from the first part of Claim 3 by induction.

CLAIM 5. *If $n \geqslant N_k$ and I is a subinterval of 2^n not contained in any of the I^n_j's, then $\pi^n_{N_k}(I) = 2^{N_k}$.*

Proof of Claim 5. Subclaim. If I is a subinterval of 2^{n+1} not contained in any of the I^{n+1}_j's and $\pi^{n+1}(I) \subseteq I^n_i$ for some i, then $\pi^{n+1}(I) = I^n_i$.

To see this, fix $s, t \in I$ such that $s \in I^{n+1}_{j_1}$, $t \in I^{n+1}_{j_2}$ with $j_1 < j_2$. Since $\pi^{n+1}(I) \subseteq I^n_i$, by Claim 3,

$$\pi^{n+1}(I^{n+1}_{j_1}) = \pi^{n+1}(I^{n+1}_{j_2}) = I^n_i.$$

Moreover, $(\max I^{n+1}_{j_1}) \,|\, n = \max I^n_i$. Since $s \leqslant^{n+1} \max I^{n+1}_{j_1} \leqslant^{n+1} t$ and I is an interval, $\max I^{n+1}_{j_1} \in I$. Therefore, $\max I^n_i \in \pi^{n+1}(I)$. Similarly, we show that $\min I^n_i \in \pi^{n+1}(I)$. It follows that $\pi^{n+1}(I) = I^n_i$.

Now it follows by induction from Subclaim and Claim 4 that $\pi^n_m(I)$, for $n \geqslant m \geqslant N_k$, is either equal to I^m_i for some $i \leqslant L_m$ or is not contained in any such I^m_i. Since $L_{N_k} = 0$ and $I^{N_k}_0 = 2^{N_k}$, this observation gives us $\pi^n_{N_k}(I) = I^{N_k}_0 = 2^{N_k}$.

CLAIM 6. *If $n > N_{k+1}$, then $\pi^n_{N_{k+1}}(I^n_i) \neq 2^{N_{k+1}}$.*

Proof of Claim 6. By Claim 4, $\pi^n_{N_{k+1}}(I^n_i) = I^{N_{k+1}}_i$ for some $i \leqslant L_{N_{k+1}}$, and this last interval is different from $2^{N_{k+1}}$ since $L_{N_{k+1}} > 0$.

Now we are ready to prove Lemma 5.3. First note that the equivalence of (i) and (ii) is simply our Claim 2. Therefore, only (ii) $\Leftrightarrow$ (iii) needs a proof.

Assume (ii) fails. Fix k. For some $m \geqslant N_k$, $[x\,|\,m, y\,|\,m]$ is not contained in any of the regular subintervals of rank i^m_k. Thus by Claim 5, $\pi^m_{N_k}([x\,|\,m, y\,|\,m]) = 2^{N_k}$. Since $N_k \geqslant k$, we obviously also have $\pi^m_k([x\,|\,m, y\,|\,m]) = 2^k$. Since k was arbitrary, the implication (ii) $\Leftarrow$ (iii) is established.

Now assume (ii). Then for some p and for all $m \geqslant n_p$, $[x\,|\,m, y\,|\,m]$ is contained in some regular subinterval of rank i^m_p. Thus, by Claim 6, for $m > n_{p+1}$, $\pi^m_{n_{p+1}}([x\,|\,m, y\,|\,m]) \neq 2^{n_{p+1}}$. This is clearly also true for $m = n_{p+1}$, so the implication $\Rightarrow$ with $k = n_{p+1}$ is proved as well.

Part 2: *Construction of Continua*

If $(I, <)$ is a finite linear order, we let $(2^n \times I, <_l)$ be the lexicographic order on $\{0, 1\} \times \cdots \{0, 1\} \times I$ with $\{0, 1\}$ taken n times and with the order $<$ on I. Again there is a natural projection $p: 2^n \times I \to I$ defined by $p(s, x) = x$ for $s \in 2^n$ and $x \in I$. For $s \in 2^n$, let I^s be $\{s\} \times I$.

Pick a sequence $(k_n) \in T$ such that for infinitely many k there exist infinitely many n with $k = k_n$. The sequence (k_n) will remain fixed from this point on. As before we write $M_n = \max\{k_i : i \leqslant n\}$.

Take now $z \subseteq 2^{<\mathbb{N}}$. We will associate with z a sequence $(I_n^z, <_n^z)$ of finite linear orders. We drop the superscript z in the description of the construction. For $s \in 2^{<\mathbb{N}}$, let $m(s) = |\{k < |s| : s \mid k \in z\}|$. For $s \in 2^n$, $k \leqslant M_{m(s \mid n-1)}$, and $i < j(s \mid n-1, k)$, we will have pairwise disjoint subintervals $K_i^{s,k}$ of I_{2n-1}. Similarly, for $s \in 2^n$, $k \leqslant M_{m(s)}$, and $i < j(s, k)$, we will have pairwise disjoint subintervals $J_i^{s,k}$ of I_{2n}. Since in each step I_n will be defined from I_{n-1} by a doubling operation or by multiplication by some 2^k, we will have the natural projection from I_n to I_{n-1}. We call it p_n.

Let the domain of $<_0$ be a one-point set I_0 and let $<_0$ be the only linear order on this domain. Note that $2^0 = \{\varnothing\}$ and $m(\varnothing) = 0$. We let $j(\varnothing, 0) = 1$ and $J_0^{\varnothing, 0} = I_0$. Assuming that $(I_{2n-2}, <_{2n-2})$ has been defined, let

$$(I_{2n-1}, <_{2n-1}) = (2^n \times I_{2n-2}, <_l).$$

Put

$$K_i^{s,k} = I_{2n-1}^s \cap p_{2n-1}^{-1}(J_i^{s \mid n-1, k}) \text{ for } i < j(s \mid n-1, k) \text{ and } k \leqslant M_{m(s \mid n-1)},$$

where p_{2n-1} is the projection from I_{2n-1} to I_{2n-2}. Having defined $(I_{2n-1}, <_{2n-1})$, let

$$\begin{aligned}(I_{2n}, <_{2n}) \\ &= D(<_{2n-1}; (K_i^{s, k_{m(s)}-1} : s \in z \cap 2^n, k_{m(s)} > 0, i < j(s \mid n-1, k_{m(s)} - 1))).\end{aligned}$$

We define $J_i^{s,k}$ for $k \leqslant M_{m(s)}$ as follows:

(i) for $k < k_{m(s)}$, let $J_j^{s,k}$, $j < 2j(s \mid n-1, k) = j(s, k)$, list all sets of the form $K_i^{s,k} \times \{0\}$ and $K_i^{s,k} \times \{1\}$;

(ii) for $k_{m(s)} \leqslant k \leqslant M_{m(s \mid n-1)}$, let $J_j^{s,k}$, $j < j(s \mid n-1, k) = j(s, k)$, list all sets of the form $I_{2n} \cap (K_i^{s,k} \times 2)$;

(iii) if $k_{m(s)} > M_{m(s \mid n-1)}$, we let $j(s, k_{m(s)}) = 1$ and define $J_0^{s, k_{m(s)}}$ to be $I_{2n} \cap (I_{2n-1}^s \times 2)$.

In the following lemma, we list some of the properties of the sets $K_i^{s,k}$ and $J_i^{s,k}$. The proof is an easy induction and we leave it to the reader.

LEMMA 5.4.

(i) *All $K_i^{s,k}$ and $J_i^{s,k}$ are intervals.*

(ii) *If $k_1 \leqslant k_2$, then for any i, j either $K_i^{s,k_1} \subseteq K_j^{s,k_2}$ or $K_i^{s,k_1} \cap K_j^{s,k_2} = \varnothing$, and similarly for J_i^{s,k_1} and J_j^{s,k_2}.*

(iii) *If* $k>0$, *then for each* i *there exists* j *with* $\min J_i^{s,k} = \min J_j^{s,k-1}$.

(iv) *If* $m(s)=0$, *then* $J_0^{s,0}$ *has precisely one element.*

For simplicity of notation, we assume that for $s \in 2^n$, $J_i^{s,k} <_{2n} J_j^{s,k}$ and $K_i^{s,k} <_{2n-1} K_j^{s,k}$ if $i<j$.

Let $z \subseteq 2^{<\mathbb{N}}$. Let $(I_n^z, <_n^z)_n$ be the sequence of finite linear orders associated with z as above. As is easy to see $|I_{2n}^z| \leqslant 2^{n+(n(n+1))/2}$ and $|I_{2n+1}^z| \leqslant 2^{1+n+(n(n+1))/2}$. Thus, we can assume that for each n, we have a finite set $\mathscr{M}_n$ of finite linear orders such that for each z, $(I_n^z, <_n^z)$ is taken from $\mathscr{M}_n$. We put the discrete topology on $\mathscr{M}_n$. Note first that the function

$$2^{2^{<\mathbb{N}}} \ni z \to (I_n^z, <_n^z)_n \in \prod_n \mathscr{M}_n$$

is continuous since each initial segment of the sequence $(I_n^z, <_n^z)_n$ depends only on $z \cap 2^m$ for some $m \in \mathbb{N}$.

Now, using inverse limits, we will continuously assign to each $(I_n^z, <_n^z)$ an indecomposable subcontinuum of $[0,1]^{\mathbb{N}}$. We fix z for the duration of the description of the continuum assigned to $(I_n^z, <_n^z)$ and, therefore, drop the superscript in I_n^z and $<_n^z$. For each $n \geqslant 1$, let $\mathscr{I}_n = [0,1]$. Fix n. Let X_n be a set with $|I_n|$ elements of equally spaced numbers in $\mathscr{I}_n$ with the additional condition that $0, 1 \in X_n$. This is possible since for $n \geqslant 1$, $|I_n| \geqslant 2$. Therefore, we can write $X_n = \{x_u : u \in I_n\}$ and assume that

$$u <_n v \text{ iff } x_u < x_v \qquad \text{for} \quad u, v \in I_n.$$

Define $f_n : \mathscr{I}_n \to \mathscr{I}_{n-1}$ to be the unique continuous function which is linear between points of the form x_u, with $u \in I_{n+1}$, and such that

$$f_n(x_u) = x_{p_n(u)}.$$

Let

$$C^z = \varprojlim(\mathscr{I}_n, f_n).$$

Let $\delta_n^m : \mathscr{I}_m \to \mathscr{I}_n$, with $m \geqslant n$, be the bonding maps and $\delta_n : C^z \to \mathscr{I}_n$ be the projection maps.

Again it is straightforward to check that the mapping $2^{2^{<\mathbb{N}}} \ni z \to C^z \in \mathscr{K}([0,1]^{\mathbb{N}})$ is continuous.

Our aim is to see that if $z \in \mathrm{IF}'$, then $\mathbb{E}_1 \sqsubseteq_c E_{C^z}$, and if $z \in \mathrm{WF}'$, then $E_{C^z} \leqslant_B \mathbb{E}_0$.

Part 3: $z \in IF' \Rightarrow \mathbb{E}_1 \sqsubseteq_c E_{C^z}$

Again since z will remain fixed, we drop the superscript z. First some preliminary definitions and lemmas. Let σ^n be a function mapping subintervals of I_n to subintervals of I_{n-1} defined by letting $\sigma^n(I) = [p_n[I]]$ for I a subinterval of I_n. We let σ_k^m be the composition $\sigma^m \circ \cdots \circ \sigma^{k+1}$. Let also σ_m^m be the identity. (This is analogous to the definition of π^n and π_k^m in Part 1.)

A sequence of intervals $J_i \subseteq I_{p_i}$, $i \in \mathbb{N}$, for some sequence of natural numbers $p_0 < p_1 < \cdots$ is called a *subsystem of* (I_n, f_n) if $\sigma_{p_i}^{p_{i+1}}(J_{i+1}) = J_i$. Such a subsystem (J_i) is called *proper* if for some i, $J_i \neq I_{p_i}$.

LEMMA 5.5. *Let k, (n_j), and $\alpha \in 2^{\mathbb{N}}$ be such that $k \leqslant M_{m(\alpha \mid n_0)}$ and for some sequence $(l_j)_j$,*

$$\sigma_{2n_j}^{2n_{j+1}}(J_{l_{j+1}}^{\alpha \mid 2n_{j+1}, k}) \cap J_{l_j}^{\alpha \mid 2n_j, k} \neq \varnothing.$$

Then the sequence $(J_{l_j}^{\alpha \mid 2n_j, k})_j$ is a subsystem of (I_n, f_n).

Proof. The lemma follows directly from Lemma 5.4(i) and the following two points which, in turn, are consequences of the definition of $K_i^{s,k}$ and $J_i^{s,k}$, and Lemma 5.4(ii).

(a) For $s \in 2^n$ with $n > N_k$, for any $J_j^{s,k}$ there exists precisely one $K_i^{s,k}$ with $p_{2n}[J_j^{s,k}] \cap K_i^{s,k} \neq \varnothing$ in which case $p_{2n}[J_j^{s,k}] = K_i^{s,k}$.

(b) For $s \in 2^n$ with $n > N_k$, for any $K_i^{s,k}$ there exists precisely one $J_i^{s \mid n-1, k}$ with $p_{2n-1}[K_j^{s,k}] \cap J_i^{s \mid n-1, k} \neq \varnothing$ in which case $p_{2n}[K_j^{s,k}] = J_i^{s \mid n-1, k}$.

Since $z \in \mathrm{IF}'$, there exists $\alpha \in 2^{\mathbb{N}}$ and a sequence $n_0 < n_1 < n_2 < \cdots$ such that $\{n: \alpha \mid n \in z\} = \{n_j: j \in \mathbb{N}\}$. Let $(2^n, <^n)$ be the sequence of finite linear orders assigned to our fixed sequence (k_n) as described in the beginning of Part 1. We define now functions $\phi_j : 2^j \to I_{2n_j}$. This function will fulfill the additional requirement

$$\psi_j[2^j] = \bigcup_{i < j(\alpha \mid n_j, 0)} J_i^{\alpha \mid n_j, 0}. \tag{4}$$

Note that $m(\alpha \mid n_0) = 0$ and $j(\alpha \mid n_0, k_0) = 1$ therefore $J_0^{\alpha \mid n_0, 0}$ is defined (and no other interval of the form $J_i^{\alpha \mid n_0, k}$ is defined) and has one element by Lemma 5.4(iv). Let $\phi_0(\varnothing)$ be the unique element of $J_0^{\alpha \mid n_0, 0}$. This defines ϕ_0 since $2^0 = \{\varnothing\}$. Assume that $\phi_j: 2^j \to I_{2n_j}$ is defined and $\phi_j[2^j] = \bigcup_{i < j(\alpha \mid n_j, 0)} J_i^{\alpha \mid n_j, 0}$. Note that for each $s \in 2^j$ there exists precisely one $u_s \in I_{2n_{j+1}-1}^{\alpha \mid n_{j+1}}$ with $\rho_{2n_j}^{2n_{j+1}-1}(u_s) = \phi_j(s)$ and $\rho_{2n-1}^{2n_{j+1}-1}(u_s) \in I_{2n-1}^{\alpha \mid n}$ for $n_j < n < n_{j+1}$. Uniqueness of this u_s follows from $\alpha \mid n \notin z$ for $n_j < n < n_{j+1}$. We actually have $u_s \in \bigcup_{i<j(\alpha|n_{j+1}-1,0)} K_i^{\alpha|n_{j+1},0}$. This implies that $(u_s, i) \in I_{2n_{j+1}}$ for $i \in 2$ and makes it possible to define $\phi_{j+1}\colon 2^{j+1} \to I_{2n_{j+1}}$ by letting

$$\phi_{j+1}(si) = (u_s, i) \qquad \text{for} \quad i \in 2.$$

Note that (4) is satisfied at this stage as well. Note further that for each j and each $t \in 2^{j+1}$, we have

$$\rho_{2n_j}^{2n_{j+1}}(\phi_{j+1}(t)) = \phi_j(t \mid j). \tag{5}$$

The two points of the following lemma are proved by a simultaneous induction.

LEMMA 5.6.

(i) *If $I \subseteq 2^j$ is a regular subinterval of rank i_k^j, then $\phi_j[I] \subseteq J_l^{\alpha|n_j,k}$ for some l.*

(ii) $s <^j t \Leftrightarrow \phi_j(s) <_j^z \phi_j(t)$ *for* $s, t \in 2^j$.

The following lemma parallels Lemma 5.3.

LEMMA 5.7. *Let $x, y \in 2^{\mathbb{N}}$. Then the following conditions are equivalent*

(i) $xE_{(k_n)}y$;

(ii) *there exist $J_i \subseteq I_{2n_i}$, $i \in \mathbb{N}$, such that $(J_i)_i$ is a proper subsystem of (I_n, f_n) and $\phi_i(x \mid n_i), \phi_i(y \mid n_i) \in J_i$ for each i;*

(iii) $\exists k\, \forall m\, (2n_m \geqslant k \Rightarrow \sigma_k^{2n_m}([\phi_m(x \mid m), \phi_m(y \mid m)]) \neq I_k)$.

Proof. (iii) $\Rightarrow$ (i) Assume that $\neg(xE_{(k_n)}y)$. Put $R_j = \phi_j[2^j]$. By (4) and Lemma 5.4(iii), we get

$$\min \bigcup_{i<j(\alpha|n_j,k),\, k \leqslant M_{m(\alpha|n_j)}} J_i^{\alpha|n_j,k} \in R_j \tag{6}$$

as well as

$$R_j \cap J_i^{\alpha|n_j,k} \neq \varnothing \text{ for all } k \leqslant M_{m(\alpha|n_j)} \text{ and } i < j(\alpha \mid n_j, k). \tag{7}$$

Fix k now. We would like to show that there exists m such that $2n_m \geqslant k$ and $\sigma_k^{2n_m}([\phi_m(x \mid m), \phi_m(y \mid m)]) = I_k$. Let n_i be such that $2n_i - 2 \geqslant k$, $\alpha \mid n_i \in z$ and $k_{m(\alpha|n_i)} > M_{m(\alpha|n_i-1)}$. This is possible by the choice of (k_n). Let $k' = k_{m(\alpha|n_i)}$. Let $j > i$ be smallest such that $k_{m(\alpha|n_j)} = k'+1$. Then $j(\alpha \mid n_j, k') \geqslant 2$, so $J_0^{\alpha|n_j,k'}$ and $J_1^{\alpha|n_j,k'}$ are defined. By (6), R_j contains an element which is $\leqslant_{2n_j} J_0^{\alpha|n_j,k'}$ and, by (7), an element of $J_1^{\alpha|n_j,k'}$. Thus $[R_j] \supseteq J_0^{\alpha|n_j,k'}$. Therefore, we have

$$\sigma_{2n_i}^{2n_j}([R_j]) \supseteq J_0^{\alpha|n_i,k'} = I_{2n_i} \cap (I_{2n_i-1}^s \times 2).$$

Thus, $\sigma_{2n_i-1}^{2n_j}([R_j]) \supseteq I_{2n_i-1}^s$ whence, since $k \leqslant 2n_i - 2$,

(8) $$\sigma_k^{2n_j}([R_j]) \supseteq I_k.$$

We assume that $\neg(xE_{(k_n)}y)$, so by Lemma 5.3, there exists $m \geqslant j$ such that $\pi_j^m([x|m, y|m]) = 2^j$. It follows from this and from (5) and Lemma 5.6(ii) that $\sigma_{2n_j}^{2n_m}([\phi_m(x|m), \phi_m(y|m)]) \supseteq R_j$, which along with (8), implies

$$\sigma_k^{2n_m}([\phi_j(x|j), \phi_j(y|j)]) = I_k.$$

(i) $\Rightarrow$ (ii) Now assume that $xE_{(k_n)}y$. By Lemma 5.3, for some k, for all $j \geqslant N_k$, $x|j$ and $y|j$ belong to the same regular subinterval of 2^j of rank i_k^j. By Lemma 5.6, for all such j, $\phi_j(x|j), \phi_j(y|j) \in J_{l_j}^{\alpha|n_j,k}$. In particular, we have

$$\rho_{2n_j}^{2n_{j+1}}[J_{l_{j+1}}^{\alpha|n_{j+1},k}] \cap J_{l_j}^{\alpha|n_j,k} \neq \varnothing.$$

By Lemma 5.5, $(J_{l_j}^{\alpha|n_j,k})_{j \geqslant N_k}$ is a subsystem of (I_n, f_n). To see its properness pick j_0 so that $k_{j_0} > k$. Then $j(\alpha|n_{j_0}, k) \geqslant 2$ and therefore there are at least two intervals of the form $J_l^{\alpha|n_{j_0},k}$ and so none of them is equal to $I_{2n_{j_0}}$.

(ii) $\Rightarrow$ (iii) is obvious. Simply let k be some n_i for which $J_i \neq I_{2n_i}$ where (J_i) is a system as in (ii).

Now we show that for our $z \in \mathrm{IF}'$, $\mathbb{E}_1 \sqsubseteq_c E_{C^z}$. By our choice of (k_n) and Lemma 4.1(iv), it will suffice to find a continuous 1-to-1 mapping $\psi: 2^{\mathbb{N}} \to C^z$ such that $xE_{(k_n)}y$ iff $\psi(x)\, E_{C^z} \psi(y)$. Define first $\psi_j: 2^j \to \mathscr{I}_{2n_j}$ by

$$\psi_j(s) = x_{\phi_j(s)}.$$

Note that for $t \in 2^{j+1}$, $\delta_{2n_j}^{2n_{j+1}}(\psi_{j+1}(t)) = \psi_j(t|j)$. It follows that the ψ_j's induce a function $\psi: 2^{\mathbb{N}} \to \varprojlim(\mathscr{I}_n, f_n)$. For $x \in 2^{\mathbb{N}}$, $\psi(x)$ is the unique $\beta \in \varprojlim(I_n, p_n)$ such that for each j, $\beta(2n_j) = \psi_j(x|j)$. Clearly, ψ is continuous and 1-to-1.

For an subinterval I of I_n, let $\hat{I}$ be the smallest subinterval of $\mathscr{I}_n$ containing all points x_u with $u \in I$. The very definition of the bonding maps f_n gives $f_n[\hat{I}] = \widehat{\sigma^n(I)}$, for any subinterval $I \subseteq I_n$, from which it follows by induction that for $m \leqslant n$,

(9) $$\delta_m^n[\hat{I}] = \widehat{\sigma^n(I)} \qquad \text{for} \quad I \subseteq I_n \text{ a subinterval.}$$

Now assume that $x, y \in 2^{\mathbb{N}}$ and $xE_{(k_n)}y$. By Lemma 5.7, we can find a proper subsystem $(J_i)_i$ of (I_n, f_n) such that $J_i \subseteq I_{2n_i}$ and $\psi_i(x|n_i), \psi_i(y|n_i) \in J_i$. Note that by the definition of subsystems and by (9), we get that

$$\delta_{2n_i}^{2n_{i+1}}[\widehat{J_{i+1}}] = \widehat{\sigma_{2n_i}^{2n_{i+1}}(J_{i+1})} = \hat{J}_i.$$

This means that $(\hat{J}_i, g_i)$, with $g_i = \delta_{2n_{i-1}}^{2n_i} | \hat{J}_i$, is an inverse system and $\varprojlim(\hat{J}_i, g_i)$ is a continuum since each $\hat{J}_i$ is a closed interval and the bonding maps are continuous. Note that, in fact, $\varprojlim(\hat{J}_i, g_i)$ is a subcontinuum of $C^z = \varprojlim(\mathscr{I}_n, f_n)$ and it is proper as $\hat{J}_i \neq I_{n_i}$, for some i, since the system (J_i) is proper. Moreover, clearly $\psi(x), \psi(y) \in \varprojlim(\hat{J}_i, g_i)$. Thus, $\psi(x)\, E_{C^z} \psi(y)$.

Now suppose that $\neg(xE_{(k_n)}y)$. Also, towards contradiction, assume that we have $\psi(x)\, E_{C^z}\psi(y)$, that is, that there exists a proper subcontinuum K of C^z which contains both $\psi(x)$ and $\psi(y)$. Then for some k,

$$\delta_k[K] \neq \mathscr{I}_k. \tag{10}$$

Fixing k from (10), Lemma 5.7 can be applied to find an m such that $2n_m \geqslant k$ and $\sigma_k^{2n_m}([\phi_m(x|m), \phi_m(y|m)]) = I_k$. By (9),

$$\delta_k^{2n_m}[\widehat{[\phi_m(x|m), \phi_m(y|m)]}] = \mathscr{I}_k. \tag{11}$$

Since $\psi(x), \psi(y) \in K$, we get

$$\delta_{n_m}[K] \supseteq \widehat{[\phi_m(x|m), \phi_m(y|m)]}. \tag{12}$$

Now (11) and (12) give $\delta_k[K] = \mathscr{I}_k$ which contradicts (10).

Part 4: $z \in WF' \Rightarrow E_{C^z} \leqslant_B \mathbb{E}_0$

Fix $z \in$ WF'. We will need some preliminary definitions and lemmas.

Let X be a set. Denote by $\mathbb{E}_0(X)$ the equivalence relation on $X^{\mathbb{N}}$ define by letting $(x_n)\, \mathbb{E}_0(X)(y_n)$ precisely when $x_n = y_n$ for large enough n. Thus, $\mathbb{E}_0(2) = \mathbb{E}_0$. Moreover, if X is countable and has at least two elements, then $\mathbb{E}_0(X) \approx_B \mathbb{E}_0$. (To see the less obvious direction $\leqslant_B$, identify X with a subset of $\mathbb{N}$ and take $X^{\mathbb{N}} \ni (x_n) \to (i_m) \in 2^{\mathbb{N}}$ with $i_m = 1$ precisely when $m = q_n^{x_n+1}$ where q_n is the n'th prime number.) If E and F are equivalence relations on sets Y and Z, let $E \times F$ be the equivalence relation on $Y \times Z$ defined by $(y_1, z_1)(E \times F)(y_2, z_2)$ iff $y_1 E y_2$ and $z_1 F z_2$. It is easy to see that $\mathbb{E}_0 \times \mathbb{E}_0 \approx_B \mathbb{E}_0$. (Simply take the mapping $2^{\mathbb{N}} \times 2^{\mathbb{N}} \notin (\alpha, \beta) \to (\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots) \in 2^{\mathbb{N}}$.) Thus, to show that for a certain equivalence relation E, $E \leqslant_B \mathbb{E}_0$, it will suffice to see that $E \leqslant_B \mathbb{E}_0(X_1) \times \mathbb{E}_0(X_2) \times \cdots \mathbb{E}_0(X_n)$ for some countable sets $X_1, \ldots, X_n$. We will frequently find this remark useful.

By 1^n and 0^n we mean the sequences in 2^n which are identically equal to 1 and 0, respectively. For $s \in 2^n \setminus \{1^n\}$, let s^+ be the successor of s in the lexicographic order $<_l$. For $s \in 2^n$ define

$$L_s^n = [\min \widehat{I_{2n-1}^s}, \min \widehat{I_{2n-1}^{s^+}}) \qquad \text{if} \quad s \neq 1^n$$

and

$$L_{1^n}^n = [\min \widehat{I_{2n-1}^{1^n}}, 1].$$

Define $g_1 \colon C^z \to (2^{<\mathbb{N}})^{\mathbb{N}}$ by putting

$$g_1(x) = (s_n^x),$$

where s_n^x is the unique element of 2^n with $x_{2n-1} \in L_{s_n^x}^n$. Let $g_2 \colon C^z \to 2^{\mathbb{N}}$ be defined by

$$g_2(x) = (i_n^x),$$

where $i_n^x = 1$ if $x_{2n-1} \in \bigcup_{s \in 2^n} \widehat{I_{2n-1}^s}$ and $i_n^x = 0$ otherwise. We easily see that g_1 and g_2 are Borel.

In the next lemma we will state two criteria useful in deciding whether or not $xE_{C^z}y$.

Lemma 5.8. *Let $x = (x_n), y = (y_n) \in C^z$.*

(i) *If $\forall^\infty n\, f_n$ is monotonic on $[x_n, y_n]$, then $xE_{c^z}y$.*

(ii) *If $\exists^\infty n\, \exists m \geqslant n\, \delta_n^m[[x_m, y_m]]$ contains 0 or 1, and also we have that $\exists^\infty n\, \exists m \geqslant n\, \delta_n^m[[0, x_m, y_m]] = \mathscr{I}_n$ and $\delta_n^m[[1, x_m, y_m]] = \mathscr{I}_n$, then $\neg(xE_{C^z}y)$.*

Proof. (i) Fix n_0 with f_n monotonic on $[x_n, y_n]$ for $n \geqslant n_0$. Monotonicity of f_n on $[x_n, y_n]$ implies $f_n[[x_n, y_n]] = [x_{n-1}, y_{n-1}]$ for $n \geqslant 1$. Note that since for infinitely many n, f_n is not monotonic on $\mathscr{I}_n$, $[x_n, y_n]$ is a proper subinterval of $\mathscr{I}_n$. Let $J_n = [x_n, y_n]$ for $n \geqslant n_0$ and $J_n = f_{n+1}[[x_{n+1}, y_{n+1}]]$ for $n < n_0$. Then $(J_n, f_n \mid J_n)$ is an inverse system, and its inverse limit is a proper subcontinuum of C^z obviously containing x and y.

(ii) Suppose that the assumption of (ii) hold and, toward contradiction, assume $xE_{C^z}y$. This places x and y in a proper subcontinuum K of C^z. By properness, there exists n with

$$\delta^n[K] \neq \mathscr{I}_n. \tag{13}$$

The assumption allows us to pick $n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4$ so that $n \leqslant n_1$, $\delta_{n_3}^{n_4}[[x_{n_4}, y_{n_4}]]$ contains 0 or 1 and $\delta_{n_1}^{n_2}[[0, x_{n_2}, y_{n_2}]]$ and $\delta_{n_1}^{n_2}[[1, x_{n_2}, y_{n_2}]]$

are both equal to $\mathscr{I}_{n_1}$. Since $\delta_l^k(0)=0$ and $\delta_l^k(1)=1$, for any $k\leqslant l$, we get that for some $i\in 2$,

$$\begin{aligned}\delta_{n_1}^{n_4}[[x_{n_4},y_{n_4}]]&=\delta_{n_1}^{n_2}\circ\delta_{n_2}^{n_3}\circ\delta_{n_3}^{n_4}[[x_{n_4},y_{n_4}]]\\&\supseteq\delta_{n_1}^{n_2}\circ\delta_{n_2}^{n_3}[[i,x_{n_3},y_{n_3}]]\supseteq\delta_{n_1}^{n_2}[[i,x_{n_2},y_{n_2}]]=\mathscr{I}_{n_1}.\end{aligned}$$

Since $n\leqslant n_1$ and since $\delta^{n_4}[K]\supseteq[x_{n_4},y_{n_4}]$, we get from this $\delta^n[K]=\mathscr{I}_n$ which contradicts (13).

We will also record some intervals on which the f_n's are monotonic. We leave checking the next lemma to the reader.

LEMMA 5.9.

(i) f_{2n-1} *is monotonic on* $\widehat{I^s_{2n-1}}$ *and* $L^n_s\setminus\widehat{I^s_{2n-1}}$ *for* $s\in 2^n$.

(ii) f_{2n} *is monotonic on* $\widehat{J_i^{s,k_{m(s)}-1}}$, *for* $s\in 2^n$ *and any* i, *and on any interval disjoint with* $\bigcup_{s\in z\cap 2^n}\bigcup_i \widehat{J_i^{s,k_{m(s)}-1}}$. *In particular, if* $s\notin z\cap 2^n$, *then* f_{2n} *is monotonic on* $f_{2n}^{-1}(L^n_s)$.

LEMMA 5.10. *For any* $z\in 2^{2^{<\mathbb{N}}}$ *and* $x,y\in C^z$, *if* $xE_{C^z}y$, *then* $g_1(x)\ \mathbb{E}_0(2^{<\mathbb{N}})\ g_1(y)$ *and* $g_2(x)\ \mathbb{E}_0\ g_2(y)$.

Proof. Let $a<b$ be two points of $\mathscr{I}_{2n-1}$. Note that if $\widehat{I^{0^n}_{2n-1}}\nless b$, then $a,b\in\widehat{I^{0^n}_{2n-1}}$, and if $a\nless\widehat{I^{1^n}_{2n-1}}$, then $a,b\in\widehat{I^{1^n}_{2n-1}}$. Thus, if a and b do not belong to one interval of the form $\widehat{I^t_{2n-1}}$ for some $t\in 2^n$, then $I^{0^n}_{2n-1}\leqslant b$ and $a\leqslant I^{1^n}_{2n-1}$, whence $f_{2n-1}[[a,b,1]]=\mathscr{I}_{2n-2}$ and $f_{2n-1}[[a,b,0]]=\mathscr{I}_{2n-2}$. In particular, for $x,y\in C^z$,

$$\text{(14)}\qquad\begin{aligned}&f_{2n-1}[[x_{2n-1},y_{2n-1},1]]=\mathscr{I}_{2n-2}&&\text{and}\\&f_{2n-1}[[x_{2n-1},y_{2n-1},0]]=\mathscr{I}_{2n-2}&&\text{if}\quad s^x_n\neq s^y_n\text{ or }i^x_n\neq i^y_n.\end{aligned}$$

Further, note that if c is an endpoint of an interval of the form $\widehat{I^s_{2n-1}}$ for some $s\in 2^n$, then $f_{2n-1}(c)$ is either 0 or 1. Now if a,b do not both belong to an interval of the form L^n_s or one of them belongs to $\bigcup_{s\in 2^n}\widehat{I^s_{2n-1}}$ and the other does not, then $[a,b]$ contains such an endpoint c. Thus, $0\in f_{2n-1}[[a,b]]$ or $1\in f_{2n-1}[[a,b]]$. It follows that for $x,y\in C^z$,

(15)

$$0\in f_{2n-1}[[x_{2n-1},y_{2n-1}]]\text{ or }1\in f_{2n-1}[[x_{2n-1},y_{2n-1}]]\text{ if }s^x_n\neq s^y_n\text{ or }i^x_n\neq i^y_n.$$

Let now $x,y\in C^z$ be such that $\neg(g_1(x)\ \mathbb{E}_0(2^{<\mathbb{N}})\ g_1(y))$ or $\neg(g_2(x)\ \mathbb{E}_0\ g_2(y))$. Then $s^x_n\neq s^y_n$ or $i^x_n\neq i^y_n$ for infinitely many n. Therefore, (14) and (15) hold for infinitely many n, whence by Lemma 5.8(ii), $\neg(xE_{C^z}y)$, and we are done.

A point $x=(x_n)\in \underleftarrow{\lim}(\mathscr{I}_n, f_n)=C^z$ is called *stable* if for all but finitely many n, if $x_{2n-1}\in L^n_s$ for some $s\in 2^n$, then $x_{2n+1}\in L^{n+1}_{s0}$ or $x_{2n+1}\in L^{n+1}_{s1}$. A point $x\in \underleftarrow{\lim}(\mathscr{I}_n, f_n)=C^z$ is *unstable* if it is not stable. Let $U=\{x\in C^z : x \text{ is unstable}\}$, and let $S=\{x\in C^z : x \text{ is stable}\}$.

LEMMA 5.11. *For any $z\in 2^{2^{<\mathbb{N}}}$, both U and S are Borel and E_{C^z}-invariant.*

Proof. Borelness of U and S follows immediately from their definitions. To show invariance of S and U, it will suffice to see that S is E_{C^z}-invariant since $U=C^z\setminus S$. Let $A=\{(s_n)\in(2^{<\mathbb{N}})^{\mathbb{N}} : \forall^\infty n\, s_{n+1}=s_n0 \text{ or } s_{n+1}=s_n1\}$. It is immediate that A is $\mathbb{E}_0(2^{<\mathbb{N}})$-invariant and $S=g_1^{-1}(A)$. Thus E_{C^z}-invariance of S follows from Lemma 5.10.

Note now that to prove $E_{C^z}\leqslant_B \mathbb{E}_0$, it is enough to show that $E_{C^z}\,|\,U\leqslant_B \mathbb{E}_0$ and $E_{C^z}\,|\,S\leqslant_B \mathbb{E}_0$. Indeed, let $g\colon U\to 2^{\mathbb{N}}$ be a Borel function witnessing the first of these inequalities and $f\colon S\to 2^{\mathbb{N}}$ be a Borel function witnessing the other one. Then the function $h\colon C^z\to 2^{\mathbb{N}}$ defined by letting $h(x)=(g(x)_0, 1, g(x)_1, 1, \ldots)$, if $x\in U$, and $h(x)=(f(x)_0, 0, f(x)_1, 0, \ldots)$, if $x\in S$, is easily seen, using Lemma 5.11, to be a witness to $E_{C^z}\leqslant_B \mathbb{E}_0$. Therefore, Lemmas 5.12 and 5.13 will finish the proof. Actually, in Lemma 5.13 we do not assume that $z\in \mathrm{WF}$' and we will use this lemma in the proof of the next theorem, as well.

LEMMA 5.12. *For any $z\in 2^{2^{<\mathbb{N}}}$, $E_{C^z}\,|\,U\leqslant_B \mathbb{E}_0$.*

Proof. We will reduce $E_{C^z}\,|\,U$ to $E'=\mathbb{E}_0(2^{<\mathbb{N}})\times\mathbb{E}_0\times\mathbb{E}_0(\mathbb{N})$. Points 0, 1 and the endpoints of the intervals $\widehat{J_i^{s,k_{m(s)}-1}}$, with $s\in z\cap 2^n$, divide $\mathscr{I}_{2n}$ into m_n intervals $[a_0,a_1), [a_1,a_2), \ldots, [a_{m_n-1}, a_{m_n}]$ where $a_0=0$ and $a_{m_n}=1$. For $x\in C^z$, let p^x_n be the unique natural number with $x_{2n}\in[a_{p^x_n}, a_{p^x_n+1})$ if $x_{2n}\notin[a_{m_n-1}, a_{m_n}]$ and $p^x_n=m_n-1$ if $x_{2n}\in[a_{m_n-1}, a_{m_n}]$. Define now

$$h(x)=(p^x_n)_n\in\mathbb{N}^{\mathbb{N}}.$$

Clearly h is Borel. We claim that the function defined for $x\in U$ by

$$f(x)=(g_1(x), g_2(x), h(x))\in(2^{<\mathbb{N}})^{\mathbb{N}}\times 2^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$$

gives us $E_{C^z}\,|\,U\leqslant_B E'$.

First, we show that if $f(x)\,E'f(y)$ for $x, y\in C^z$, then $xE_{C^z}y$. Our assumption allows us to find n_0 such that $i^x_n=i^y_n$, $s^x_n=s^y_n$, $p^x_n=p^y_n$ for all $n\geqslant n_0$. We will show that for $m\geqslant 2n_0+1$, f_m is monotonic on $[x_m, y_m]$, and we will be done by Lemma 5.8(i). When $m=2n-1$, $s^x_n=s^y_n$ and

$i_n^x = i_n^y$, so $[x_m, y_m]$ is included in one of the intervals $\widehat{I_n^s}$ or $L_s^n \setminus \widehat{I_n^s}$ with $s = s_n^x = s_n^y$, and f_m is monotonic on both these intervals by Lemma 5.9(i). If $m = 2n$, $p_n^x = p_n^y$ implies that $[x_m, y_m]$ is included in one interval of the form $[a_i, a_{i+1})$ or $[a_{m_n-1}, a_{m_n}]$, and f_m is monotonic on each such interval by Lemma 5.9(ii).

To finish the proof, it remains to see that if $x, y \in U$ and $\neg(f(x)\, E' f(y))$, then $\neg(x E_{C^z}\, y)$. If $\neg(g_1(x)\, \mathbb{E}_0(2^{<\mathbb{N}})\, g_1(y))$ or $\neg(g_2(x)\, \mathbb{E}_0\, g_2(y))$, then we get $\neg(x E_{C^z}\, y)$ by Lemma 5.10. Thus, we assume that $g_1(x)\, \mathbb{E}_0(2^{<\mathbb{N}})\, g_1(y)$, $g_2(x)\, \mathbb{E}_0\, g_2(y)$, and $\neg(h(x)\, \mathbb{E}_0(\mathbb{N})\, h(y))$. Again we find n_0 such that $s_n^x = s_n^y$ and $i_n^x = i_n^y$, for all $n \geqslant n_0$. For such n we will write s_n and i_n for the common value of these parameters.

CLAIM 1. *Assume we have two numbers n_1, n_2 with the following properties:*

(i) $n_0 \leqslant n_1 \leqslant n_2$;

(ii) $s_{n_1} \mid n_1 - 1 \neq s_{n_1 - 1}$;

(iii) $p_{n_2}^x \neq p_{n_2}^y$;

(iv) $s_n \mid n - 1 = s_{n-1}$ *for* $n_1 < n \leqslant n_2$;

(v) $p_n^x = p_n^y$ *for* $n_1 \leqslant n < n_2$.

Put $k' = k_{m(s_{n_2})} - 1$. *Then*

(vi) $k' > M_{m(s_{n_1} \mid n_1 - 1)}$;

(vii) $[x_{2n_2}, y_{2n_2}]$ *contains an endpoint of an interval of the form* $\widehat{J_i^{s_{n_2}, k'}}$;

(viii) $[0, x_{2n_2}, y_{2n_2}]$ *and* $[1, x_{2n_2}, y_{2n_2}]$ *each contains an interval of the form* $\widehat{J_i^{s_{n_2}, k'}}$.

Proof of Claim 1. Subclaim. $[x_{2n_1-1}, y_{2n_1-1}]$ does not intersect any interval of the form $\widehat{K_i^{s,k}}$ for $s \in 2^{n_1}$ and $k \leqslant M_{m(s \mid n_1 - 1)}$.

Proof of Subclaim. $[x_{2n_1-1}, y_{2n_1-1}]$ is included in $L_{s_{n_1}}^{n_1}$ so it does not intersect any $\widehat{K_i^{s,k}}$ for $s \in 2^{n_1}$ with $s \neq s_{n_1}$ since each such interval is included in $L_s^{n_1}$. $[x_{2n_1-3}, y_{2n_1-3}]$ is included in $L_{s_{n_1-1}}^{n_1-1}$ and $\widehat{K_i^{s_{n_1} \mid n_1-1, k}}$ in $L_{s_{n_1} \mid n_1 - 1}^{n_1-1}$. Thus

$$[x_{2n_1-1}, y_{2n_1-1}] \subseteq (\delta_{2n_1-3}^{2n_1-1})^{-1}\,(L_{s_{n_1-1}}^{n_1-1})$$

and

$$\widehat{K_i^{s_{n_1}, k}} \subseteq (\delta_{2n_1-3}^{2n_1-1})^{-1}\,(L_{s_{n_1} \mid n_1 - 1}^{n_1-1}).$$

Since $s_{n_1} \mid n_1 - 1 \neq s_{n_1-1}$, we see that $L^{n_1-1}_{s_{n_1-1}}$ and $L^{n_1-1}_{s_{n_1} \mid n_1-1}$ are disjoint. Therefore, $[x_{2n_1-1}, y_{2n_1-1}]$ is disjoint from each interval $\widehat{K_i^{s_{n_1}, k}}$, as well, and the subclaim is established.

Since $s^x_n = s^y_n$, $i^x_n = i^y_n$, for $n \geqslant n_1$, and $p^x_n = p^y_n$, for $n_1 \leqslant n < n_2$, as in the first part of the proof of this lemma, we see that f_m is monotonic on $[x_m, y_m]$ for $2n_1 - 1 \leqslant m < 2n_2$. In particular, $f_m[[x_m, y_m]] = [x_{m-1}, y_{m-1}]$ for $2n_1 - 1 \leqslant m < 2n_2$. From this, Subclaim, and the easy to see inclusions, for $s \in 2^n$ and $k \leqslant M_{m(s)}$,

$$\bigcup_i \widehat{K_i^{s0,k}} \cup \widehat{K_i^{s1,k}} \subseteq f^{-1}_{2n-1}\left(\bigcup_i \widehat{J_i^{s,k}}\right)$$

and

$$\bigcup_i \widehat{J_i^{s,k}} \subseteq f^{-1}_{2n}\left(\bigcup_i \widehat{K_i^{s,k}}\right), \tag{16}$$

we get, by induction, that $[x_{2n-1}, y_{2n-1}]$, for $n_1 \leqslant n \leqslant n_2$, and $[x_{2n}, y_{2n}]$, for $n_1 \leqslant n < n_2$, do not intersect any interval of the form $\widehat{K_i^{s,k}}$ and $\widehat{J_i^{s,k}}$ for $s \in 2^n$ and $k \leqslant M_{m(s \mid n_1 - 1)}$. In particular, $[x_{2n_2-1}, y_{2n_2-1}]$ is disjoint from $\widehat{K_i^{s,k}}$ for $s \in 2^{n_2}$ and $k \leqslant M_{m(s \mid n_1-1)}$.

To see (vi), assume towards contradiction that it fails. $k' = k_{m(s_{n_2})} - 1 \leqslant M_{m(s_{n_1} \mid n_1 - 1)}$. Then since $[x_{2n_2-1}, y_{2n_2-1}]$ is disjoint from $\widehat{K_i^{s,k'}}$ for $s \in 2^{n_2}$, we get that f_{2n_2} is monotonic on $[x_{2n_2}, y_{2n_2}]$ by Lemma 5.9(ii), so $f_{n_2}[[x_{2n_2}, y_{2n_2}]] = [x_{2n_2-1}, y_{2n_2-1}]$. This, along with (16) and the fact that $[x_{2n_2-1}, y_{2n_2-1}]$ is disjoint from $\widehat{K_i^{s,k'}}$ for $s \in 2^{n_2}$, implies that $[x_{2n_2}, y_{2n_2}]$ does not contain an endpoint of an interval of the form $\widehat{J_i^{s,k'}}$ for $s \in 2^{n_2}$ which means that $p^x_{n_2} = p^y_{n_2}$. This contradicts our choice of n_2.

To see (vii), note that $[x_{2n_2}, y_{2n_2}]$ is included in $\overline{I_{2n_2} \cap (2 \times I^{s_{n_2}}_{2n_2-1})}$. Thus, $[x_{2n_2}, y_{2n_2}]$ does not intersect any interval $\widehat{J_i^{s,k}}$ with $s \in 2^{n_2}$ and $s \neq s_{n_2}$. Therefore, if (vii) failed, we would directly get $p^x_{n_2} = p^y_{n_2}$, contradiction. If (viii) failed, there would be no interval of the form $\widehat{J_i^{s_{n_2},k'}}$ which is $< \max\{x_{2n_2}, y_{2n_2}\}$ or which is $> \min\{x_{2n_2}, y_{2n_2}\}$. In either case, $p^x_{n_2} = p^y_{n_2}$, contradiction. Claim 1 is established.

Claim 2. *Assume n_1, n_2 fulfill* (i)–(v) *of Claim* 1. *Then*

(i) $\delta^{2n_2}_{2n_1-1}[[x_{2n_2}, y_{2n_2}]]$ *contains* 0 *or* 1;

(ii) $\delta^{2n_2}_{2n_1-1}[[0, x_{2n_2}, y_{2n_2}]] = \delta^{2n_2}_{2n_1-1}[[1, x_{2n_2}, y_{2n_2}]] = \mathscr{I}_{2n_1-1}$.

Proof of Claim 2. Let, as in Claim 1, $k' = k_{m(s_{n_2})} - 1$. Let $s \subseteq s_{n_2}$ be shortest with $k_{m(s)} = k'$. By Claim 1(vi), $s = s_m$ with $n_1 \leqslant m < n_2$. Note that $k_{m(s_m)} > M_{m(s_m \mid m-1)}$, whence for any i

$$\delta_{2m}^{2n_2}[\widehat{J_i^{s_{n_2}, k'}}] = \widehat{J_0^{s_m, k'}} = \widehat{I_{2m} \cap (I_{2m-1}^{s_m} \times 2)}.$$

Thus,

$$\delta_{2m-1}^{2n_2}[\widehat{J_i^{s_{n_2}, k'}}] = \mathscr{I}_{2m-1}.$$

Now (ii) follows immediately from Claim 1(viii) and $m \geqslant n_1$. Point (i) is also immediate from Claim 1(vii) if we only notice that $\delta_{2n_1-1}^{2n_2}$ maps endpoints of $\widehat{J_i^{s_{n_2}, k'}}$ to endpoints of $\mathscr{I}_{2n_1-1}$. This finishes the proof of Claim 2.

To see $\neg(xE_{C^z}y)$, by Lemma 5.8(ii) and Claim 2, it will suffice to find for any m^0 a pair of numbers n_1, n_2 fulfilling (i)–(v) from Claim 1 and $m^0 \leqslant n_1 \leqslant n_2$. Therefore, let us fix m^0. Let $m_0 < m_1 < m_2 < \cdots$ list all $m \geqslant m^0$ with $s_m \mid m-1 \neq s_{m-1}$. (We use here the assumption $x \in U$.) Since $p_n^x \neq p_n^y$ for infinitely many n, we can find n and i so that $m_i \leqslant n < m_{i+1}$ and $p_n^x \neq p_n^y$. Let $n_1 = m_i$ and let n_2 be the minimal n with $m_i \leqslant n$ and $p_n^x \neq p_n^y$. It is easy to check that n_1 and n_2 are as required, and Lemma 5.12 is proved.

LEMMA 5.13. *Let $z \in WF'$. Then $E_{C^z} \mid S \leqslant_B \mathbb{E}_0$.*

Proof. Define $f\colon C^z \to (2^{<\mathbb{N}})^{\mathbb{N}} \times 2^{\mathbb{N}}$ by $f(x) = (g_1(x), g_2(x))$. The function f is Borel as g_1 and g_2 are and by Lemma 5.10, if $x, y \in C^z$ and $xE_{C^z}y$, then $f(x)(\mathbb{E}_0(2^{<\mathbb{N}}) \times \mathbb{E}_0)\, f(y)$. We claim that if $x, y \in S$, then also $f(x)(\mathbb{E}_0(2^{<\mathbb{N}}) \times \mathbb{E}_0)\, f(y)$ implies $xE_{C^z}y$.

If $x \in S$, then, for large enough n, $s_n^x \subseteq s_{n+1}^x$. Therefore, if $z \in WF'$, then, for all but finitely many n, $s_n^x \notin z$.

Assume now $x, y \in S$ and $f(x)(\mathbb{E}_0(2^{<\mathbb{N}}) \times \mathbb{E}_0)\, f(y)$. It follows from the above argument that we can fix n_0 such that

$$s_n^x = s_n^y \notin z \text{ and } i_n^x = i_n^y \text{ for } n \geqslant n_0. \tag{17}$$

Let now $s_n = s_n^x = s_n^y$ for $n \geqslant n_0$ and assume, as we can, $n_0 \geqslant 1$. By (17), $x_{2n-1}, y_{2n-1} \in \widehat{I_{2n-1}^{s_n}}$ or $x_{2n-1}, y_{2n-1} \in L_s^n \setminus \widehat{I_{2n-1}^{s}}$ for $n \geqslant n_0$. Thus, for such n, $f_{2n-1} \mid [x_{2n-1}, y_{2n-1}]$ is monotonic by Lemma 5.9(i). Since $s_n \notin z$ for $n \geqslant n_0$, $f_{2n} \mid [x_{2n}, y_{2n}]$ is monotonic by Lemma 5.9(ii) since $[x_{2n}, y_{2n}] \subseteq f_{2n}^{-1}(L_s^n)$. An application of Lemma 5.8(i) finishes the proof of Lemma 5.13.

Thus Theorem 5.2 is established as well.

The following corollary follows directly from Theorems 5.1 and 5.2.

COROLLARY 5.14. *The set*

$$\{C \in \mathscr{K}([0,1]^{\mathbb{N}}) : C \text{ is an indecomposable continuum and } E_C \approx_B \mathbb{E}_1\}$$

is Σ^1_1*-complete*

One may ask if $E_C \approx_B \mathbb{E}_1$ implies that E_C behaves similarly to $\mathbb{E}_1$ on a large (nonmeager) subset of C. Also, as explained in the introduction (1.3), by [M, Theorem 7.2], it is natural to view Kuratowski's problem on generic ergodicity of the composant equivalence relation as an extension of the problem, solved in this paper, on the existence of Borel transversal for the set of all composant. One may try to solve the former problem by extending the solution to the latter by proving that the conditions $E_C \approx_B \mathbb{E}_1$ and $E_C \approx_B \mathbb{E}_0$ are equivalent to the existence of a Borel isomorphism preserving meagerness between E_C and $\mathbb{E}_1$ or $\mathbb{E}_0 \times 2^{\mathbb{N}}$, respectively. This would do, since $\mathbb{E}_1$ and $\mathbb{E}_0 \times 2^{\mathbb{N}}$ are generically ergodic. The next theorem and its corollary address the above two questions and show that for some of the indecomposable continua C constructed in Theorem 5.2, $E_C \approx_B \mathbb{E}_1$ may hold because of a "wild behavior" of the composants only on a small (meager) subset of C and so there exist indecomposable continua C for which there is no Borel isomorphism between E_C and $\mathbb{E}_1$ or $\mathbb{E}_0 \times 2^{\mathbb{N}}$ which preserves meager sets. See, however, question 1 in Section 6.

We will need the following lemma.

LEMMA 5.15. *Let* $z \in 2^{2^{<\mathbb{N}}}$. *The set of all unstable points in* C^z *is comeager.*

Proof. As before U stands for the set of all unstable points. Note that U contains the set of $x = (x_n) \in C^z$ described by the following condition (int and cl stand for interior and closure in the interval $[0, 1]$)

$$\forall m\, \exists n > m\, x_{2n-1} \in \operatorname{int}(L^n_s) \text{ and } x_{2n+1} \notin \operatorname{cl}(L^{n+1}_{s0}) \text{ and } x_{2n+1} \notin \operatorname{cl}(L^{n+1}_{s1}).$$

But, as is easy to see, for any given m, the set of all (x_n) fulfilling

$$\exists n > m\, x_{2n-1} \in \operatorname{int}(L^n_s) \text{ and } x_{2n+1} \notin \operatorname{cl}(L^{n+1}_{s0}) \text{ and } x_{2n+1} \notin \operatorname{cl}(L^{n+1}_{s1})$$

is open and dense in C^z, so we are done.

THEOREM 5.16. *There exists an indecomposable continuum* C *such that*

(i) $\mathbb{E}_1 \sqsubseteq_c E_C$ *so* $E_C \approx_B \mathbb{E}_1$ *and*

(ii) $E_C \mid U \leqslant_B \mathbb{E}_0$ *for a comeager* E_C*-invariant subset* U *of* C.

Proof. Pick any $z \in$ IF'. Let $C = C^z$. By Theorem 5.2, we have (i). Let U be the set of all unstable points in C. From Lemmas 5.11, 5.12, and 5.15, we get (ii).

Corollary 5.17 follows immediately from Theorem 5.16 and the fact that if D is an $\mathbb{E}_1$-invariant, comeager subset of $(2^{\mathbb{N}})^{\mathbb{N}}$, then $\mathbb{E}_1 \leqslant_B \mathbb{E}_1 \mid D$.

COROLLARY 5.17. *There exists an indecomposable continuum C for which $E_C \approx_B \mathbb{E}_1$, but there is no Borel isomorphism between E_C and $\mathbb{E}_1$ which preserves meager sets.*

6. OPEN QUESTIONS

1. Is it true that for each indecomposable continuum C, one can find an E_C-invariant, comeager set $D \subseteq C$ such that $E_C \mid D$ is isomorphic to $\mathbb{E}_1$ or $\mathbb{E}_0 \times 2^{\mathbb{N}}$ via a Borel isomorphism preserving meager sets?

Note that 1 is not ruled out by Corollary 5.17. I checked that Knaster continua have the property from question 1. An affirmative answer to this question would imply an affirmative answer to the following old problem of Kuratowski.

2. (Kuratowski [Ku, p. 255]) Let C be an indecomposable continuum. Is E_C generically ergodic? That is, is each Borel subset of C which is the union of a family of composants either meager or comeager?

It was proved that this question has an affirmative answer for the Knaster buckethandle continuum [Ku], Knaster continua, solenoids, and the pseudo-arc [Em], and for a class of continua of unknown extent (but certainly containing the buckethandle continuum and solenoids) called simple continua [Kr].

Note that the following weaker version of 1 would still imply a positive solution to 2.

1'. Let C be an indecomposable continuum. Does the following hold? There exists a Borel function $f_1 \colon 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to C$ for which

(a) images of comeager sets are comeager;

(b) if $x(\mathbb{E}_0 \times 2^{\mathbb{N}})\, y$, then $f_1(x)\, E_C\, f_1(y)$;

or there exists a Borel function $f_2 \colon (2^{\mathbb{N}})^{\mathbb{N}} \to C$ satisfying properties (a) and (b) with $\mathbb{E}_0 \times 2^{\mathbb{N}}$ in (b) replaced by $\mathbb{E}_1$.

3. Can one find a classical (that is, not using effective descriptive set theory) proof of Theorem 5.1?

An affirmative answer to 3 may give a new characterization of the condition $E_C \approx_B \mathbb{E}_1$.

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